The 43rd Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: February 24, 2024 at Regional Testing Sites

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COMPREHENSIVE EXAMINATION

Construction of this test directed
by
Ashley Johnson, University of North Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E. A calculator is NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

Editing by Miranda Bowie and Ashley Johnson, The University of North Alabama
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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Find the sum of all values of x which minimize the function $f(x) = |x^3 + 4x^2 - 3x - 12|$.

(**A**) $\boxed{-4}$ (**B**) 4 (**C**) $-4 + \sqrt{3}$ (**D**) $4 + \sqrt{3}$ (**E**) None of these

2. If $f(1) = 3$ and $f(4) = -2$ for $f(x) = \dfrac{x}{b} + \dfrac{a}{b}$, find $\dfrac{a}{b}$.

(**A**) $-\dfrac{1}{3}$ (**B**) $\dfrac{2}{5}$ (**C**) $\boxed{\dfrac{14}{3}}$ (**D**) $\dfrac{18}{5}$ (**E**) None of these

3. Define $f(x) = \begin{cases} x + 9 & \text{for } x < 4 \\ x^2 - 3 & \text{for } 4 \le x < 6 \\ 5x + 3 & \text{for } x \ge 6 \end{cases}$. Find $f^{-1}(15)$.

(**A**) $\dfrac{1}{78}$ (**B**) $\dfrac{12}{5}$ (**C**) 6 (**D**) $\boxed{3\sqrt{2}}$ (**E**) None of these

4. The graph of $|2x| + |y| = 4$ encloses a region of the plane. What is the area of the region?

(**A**) $\boxed{16}$ (**B**) 20 (**C**) 32 (**D**) 40 (**E**) None of these

5. What is the number of subsets of the set $\{a, b, c, d, e, f\}$ that contain at least one vowel?

(**A**) 20 (**B**) 32 (**C**) $\boxed{48}$ (**D**) 64 (**E**) None of these

6. Define an operation $\circledast$ on pairs of real numbers as

$$(a, b) \circledast (c, d) = (ac - bd, bc + ad).$$

If $(1, -3) \circledast (c, 6) = \left(x, \frac{3}{2}\right)$, find x.

(**A**) $\dfrac{3}{2}$ (**B**) $\dfrac{31}{2}$ (**C**) $\dfrac{37}{2}$ (**D**) $\boxed{\dfrac{39}{2}}$ (**E**) None of these

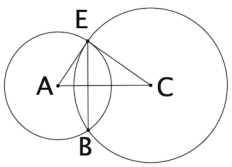
7. How many different three digit integers have the sum of their three digits equal to 18 and are divisible by 5?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) $\boxed{7}$ (**E**) None of these

8. If a and b are the two solutions to $(2x + 3)(3x - 1) = -4$, find $a^2 + b^2$.

(**A**) $\dfrac{53}{4}$ (**B**) $\dfrac{29}{4}$ (**C**) $\dfrac{13}{36}$ (**D**) $\boxed{\dfrac{37}{36}}$ (**E**) None of these

9. In the figure shown, the circles centered at A and C intersect at points E and B. If $\overline{EB}$ is perpendicular to $\overline{AC}$, $AE = 26$, $AC = 28$, and $CE = 30$, find BE.



(**A**) 10 (**B**) 20 (**C**) 24 (**D**) $\boxed{48}$ (**E**) None of these

10. How many distinct, real zeros does the function $f(x) = x^5 + x^4 + 4x^3 + 6x^2 - 12x$ have?

(**A**) 2 (**B**) $\boxed{3}$ (**C**) 4 (**D**) 5 (**E**) None of these

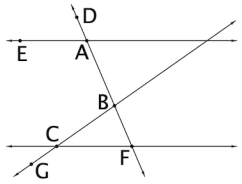
11. A right triangle has a perimeter of 32 units and an area of 20 square units. Find the length of the hypotenuse.

(**A**) $\frac{26}{3}$ (**B**) $\boxed{\frac{59}{4}}$ (**C**) $\frac{69}{4}$ (**D**) $\frac{70}{3}$ (**E**) None of these

12. Which of the following pairs is a solution to the equation $x^2 - 3y + x^2y - 3 = 0$?

(**A**) $(1-\sqrt{7}, \sqrt{3})$ (**B**) $(2+\sqrt{11}, \sqrt{5})$ (**C**) $\boxed{(\sqrt{3}, 1-\sqrt{7})}$ (**D**) $(\sqrt{5}, 2+\sqrt{11})$ (**E**) None of these

13. In the diagram shown, G lies on $\overleftrightarrow{BC}$, B and D lie on $\overleftrightarrow{AF}$, $\overleftrightarrow{EA} \parallel \overleftrightarrow{CF}$, $m\angle EAD = 32°$, and $m\angle ABC = 104°$. Find $m\angle FCG$.



(**A**) 72° (**B**) $\boxed{108°}$ (**C**) 136° (**D**) 148° (**E**) None of these

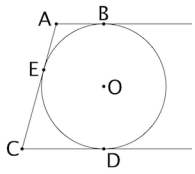
14. Find the horizontal asymptote of $y = \frac{4}{x} - 3 - \frac{x-1}{x^2-5x}$.

(**A**) $y = -4$ (**B**) $y = 1$ (**C**) $y = 4$ (**D**) $y = 5$ (**E**) $\boxed{\text{None of these: } y = -3}$

15. Let x and y be non-negative angle measures in degrees. If $\sin(x+y) = 1$, and $\cos(x-y) = \frac{1}{2}$, what is the smallest possible value of xy?

(**A**) 0 (**B**) $\boxed{1125}$ (**C**) 1800 (**D**) $\frac{6075}{4}$ (**E**) None of these

16. In the diagram shown, $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{CD}$ are tangent to the circle at points B, E, and D, respectively. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $AE = 4$, and $EC = 9$, what is the radius of the circle?



(**A**) $\frac{5}{2}$ (**B**) $2\sqrt{5}$ (**C**) $\sqrt{65}$ (**D**) $\boxed{6}$ (**E**) None of these

17. Evaluate the expression $\frac{y^2+5y+3}{y^3+7y^2+13y+6}$ at $y = \sqrt{5}+1$.

(**A**) $\boxed{-\frac{\sqrt{5}}{4} + \frac{3}{4}}$ (**B**) $-\frac{\sqrt{5}}{5} - \frac{1}{3}$ (**C**) $\frac{\sqrt{5}}{4} - \frac{3}{4}$ (**D**) $\frac{\sqrt{5}}{5} + \frac{1}{3}$ (**E**) None of these

18. The inequality $\frac{x^2-2x+1}{x-3} < 10$ holds for the intervals $(-\infty, 3)$ and (a, b). Find $a+b$.

(**A**) 10 (**B**) $\boxed{12}$ (**C**) $2\sqrt{5}$ (**D**) $6\sqrt{5}$ (**E**) None of these

19. What is the exact value of $\sin^2 1° + \sin^2 2° + \sin^2 3° + \cdots \sin^2 90°$?

(**A**) 44 (**B**) 44.5 (**C**) 45 (**D**) $\boxed{45.5}$ (**E**) None of these

20. A polynomial $p(x)$ has a remainder of 4 when divided by $x+2$ and a remainder of 14 when divided by $x-3$. What is the remainder when $p(x)$ is divided by $(x+2)(x-3)$?

(**A**) 18 (**B**) 56 (**C**) $2x-6$ (**D**) $\boxed{2x+8}$ (**E**) None of these

21. Find the largest possible value of $x+y+z$ for x, y, and z satisfying the system of equations

$$\begin{cases} 12x^2+12xy+12xz &= 35 \\ 9y^2+9yz+9xy &= -35 \\ z^2+xz+yz &= 35 \end{cases}$$

(**A**) $\frac{35}{3}$ (**B**) $\frac{35}{4}$ (**C**) $\boxed{\frac{35}{6}}$ (**D**) $\frac{35}{12}$ (**E**) None of these

22. If $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$, what is the value of y when $x=\sqrt{71}$?

(**A**) 6 (**B**) $\boxed{9}$ (**C**) $\sqrt{71}$ (**D**) 71 (**E**) None of these

23. The function $y=x^2-20x+9$ can be written as $y=(x+a)^2+b$. Find $a+b$.

(**A**) $\boxed{-101}$ (**B**) -81 (**C**) -1 (**D**) 19 (**E**) None of these

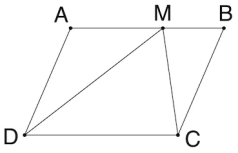
24. A box of chocolates has 10 pieces of chocolate that look identical, with four filled with caramel, four filled with orange cream, and two solid chocolate. If two are selected randomly without replacement, what is the probability at least one is filled with caramel?

(**A**) $\boxed{\frac{2}{3}}$ (**B**) $\frac{2}{5}$ (**C**) $\frac{3}{5}$ (**D**) $\frac{16}{25}$ (**E**) None of these

25. For which positive value of h does the equation $2x^2+20x+h^2=13$ have one solution which is four times the other?

(**A**) $\sqrt{3}$ (**B**) $\sqrt{19}$ (**C**) $\boxed{3\sqrt{5}}$ (**D**) $4\sqrt{2}$ (**E**) None of these

26. Let $ABCD$ be a parallelogram and M be a point on the segment $\overline{AB}$ such that the area of $\triangle MBC = 8$ and the area of $\triangle MDC = 20$. Find the area of parallelogram $ABCD$.



(**A**) 32 (**B**) 36 (**C**) $\boxed{40}$ (**D**) 44 (**E**) None of these

27. For $f(x)=x^5-10x^4-40x^3+15x^2-28x+19$, evaluate $f(13)$.

(**A**) -39 (**B**) $\boxed{-7}$ (**C**) 0 (**D**) 19 (**E**) None of these

28. Let A and B be the points of intersection of the circles $x^2+y^2-6x+4y=3$ and $x^2+y^2+4x-4y=17$. Determine the slope of the segment $\overline{AB}$.

(**A**) $\boxed{\frac{5}{4}}$ (**B**) $\frac{4}{5}$ (**C**) $-\frac{5}{4}$ (**D**) $-\frac{4}{5}$ (**E**) None of these

29. Find the sum of all real solutions to the equation

$$\frac{2-x}{x^2}+\frac{3x^2}{x-2}=2.$$

(**A**) $\boxed{-\frac{1}{3}}$ (**B**) $\frac{2}{3}$ (**C**) $-\frac{4}{3}$ (**D**) $\frac{5}{3}$ (**E**) None of these

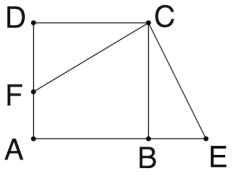
30. Which of the following is not in the range of $f(x)=x^2-2x^4+6$?

(**A**) $\sqrt{17}$ (**B**) $\sqrt{29}$ (**C**) $\sqrt{35}$ (**D**) $\boxed{\sqrt{43}}$ (**E**) None of these

31. Let $f(x)=ax^7+bx^3+cx-7$, where a, b, and c are real constants. If $f(-5)=5$, what is the value of $f(5)$?

(**A**) $\boxed{-19}$ (**B**) -5 (**C**) 5 (**D**) 13 (**E**) None of these

32. In square $ABCD$, F is a point on side $\overline{AD}$ and E is a point on the extension of side $\overline{AB}$ such that $\overline{CF}$ and $\overline{CE}$ are perpendicular. If each side of the square has a length of 20 and the area of triangle $\triangle CEF$ is 288, what is the area of triangle $\triangle AFE$?



(**A**) 56 (**B**) 72 (**C**) $\boxed{112}$ (**D**) 144 (**E**) None of these

33. Five test scores have a mean of 91, a median of 92, and a mode of 95. Find the sum of the two lowest test scores.

(**A**) 177 (**B**) $\boxed{173}$ (**C**) 171 (**D**) 168 (**E**) None of these

34. If x and y are real numbers, determine the number of solutions in the following system of equations.

$$\begin{cases} x^2-xy+8 &= 0 \\ x^2-8x+y &= 0 \end{cases}$$

(**A**) 1 (**B**) 2 (**C**) $\boxed{3}$ (**D**) 4 (**E**) None of these

35. Find the sum of the solutions of the following equation:

$$\log_3(9^x+6)-\log_3(4\cdot 3^x-7)=1.$$

(**A**) $\boxed{3}$ (**B**) 4 (**C**) 9 (**D**) 12 (**E**) None of these

36. The perimeters of a regular hexagon and a square are equal. If α is the area of the hexagon and β is the area of the square, what is α/β?

(**A**) $\sqrt{3}$ (**B**) $\frac{\sqrt{3}}{2}$ (**C**) $\frac{3\sqrt{3}}{2}$ (**D**) $\boxed{\frac{2\sqrt{3}}{3}}$ (**E**) None of these

37. Which of the following is a solution to $(x^2+6x-3)(x^2+6x)=10$?

(**A**) $-3+\sqrt{22}$ (**B**) $-3+\sqrt{19}$ (**C**) $-3+\sqrt{17}$ (**D**) $-3+\sqrt{11}$ (**E**) $\boxed{\text{None of these}}$

38. What is the maximum value of the function $f(x) = \dfrac{3x^2+9x+11}{x^2+3x+3}$?

(**A**) 3 (**B**) $\dfrac{11}{3}$ (**C**) $\boxed{\dfrac{17}{3}}$ (**D**) $\dfrac{125}{39}$ (**E**) None of these

39. Recall that the absolute value of a complex number $a+bi$ is defined as $|a+bi| = \sqrt{a^2+b^2}$. Find $|\sqrt{3-4i}|$.

(**A**) $\sqrt{3}$ (**B**) $\boxed{\sqrt{5}}$ (**C**) $\sqrt{7}$ (**D**) 5 (**E**) None of these

40. Two positive numbers differ by 5 while their squares differ by 865. What is the sum of the two numbers?

(**A**) 145 (**B**) 157 (**C**) 161 (**D**) $\boxed{173}$ (**E**) None of these

41. A right prism with equilateral triangle bases has a volume of $120\sqrt{3}$. If the distance between bases is 10, what is the surface area of the prism?

(**A**) $\boxed{144\sqrt{3}}$ (**B**) $204\sqrt{3}$ (**C**) $24\sqrt{3}+120$ (**D**) $24\sqrt{3}+180$ (**E**) None of these

42. Let A, B, and C be on a circle such that O lies on $\overline{AC}$. If $AB = 36$, and $BC = 15$, what is the length of arc $\widehat{AC}$?

(**A**) $\boxed{\dfrac{39}{2}\pi}$ (**B**) $\dfrac{41}{2}\pi$ (**C**) 78π (**D**) 82π (**E**) None of these

43. How many solutions does the equation $\cos x - \sin^2 x \cos x = 1$ have on the interval $[0, 2\pi)$?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

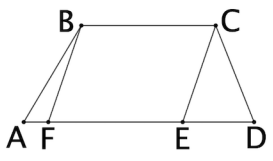
44. A stack of game cards contains 4 green, 5 blue, 3 yellow, and 3 red cards. The player must draw until they get a blue card, and then they stop. What is the probability they draw a blue card within the first three cards? Assume these cards are drawn without replacement.

(**A**) $\dfrac{24}{91}$ (**B**) $\dfrac{59}{91}$ (**C**) $\boxed{\dfrac{67}{91}}$ (**D**) $\dfrac{73}{91}$ (**E**) None of these

45. Find the median of all solutions to the equation $(\sin x + \cos x)^2 = \dfrac{1}{2}$ on the interval $[0, 2\pi)$.

(**A**) $\dfrac{3\pi}{4}$ (**B**) $\boxed{\dfrac{5\pi}{4}}$ (**C**) $\dfrac{3\pi}{2}$ (**D**) $\dfrac{5\pi}{2}$ (**E**) None of these

46. A parallelogram $\square BCEF$ is inscribed into trapezoid $ABCD$ as shown in the figure with $EC = CD$. If $AF = 4$, $BC = 10$, $ED = 12$, and $m\angle BAF = 30°$, find the area of $\square ABCD$.



(**A**) $\dfrac{104\sqrt{3}}{2}$ (**B**) $\dfrac{260\sqrt{3}}{3}$ (**C**) $24\sqrt{3}$ (**D**) $\boxed{60\sqrt{3}}$ (**E**) None of these

47. What is the largest value of a for which the point (a, b) lies on the circle $(x-4)^2 + (y+1)^2 = 9$?

(**A**) -1 (**B**) 5 (**C**) $\boxed{7}$ (**D**) 13 (**E**) None of these

48. Two solutions to the equation $x^4 - 6x^3 - 37x^2 + 178x + 44 = 0$ are $2 + \sqrt{5}$ and $1 - 3\sqrt{5}$. Find the sum of the other two.

(**A**) $-3 - 2\sqrt{5}$ (**B**) $-1 - 4\sqrt{5}$ (**C**) $1 + 4\sqrt{5}$ (**D**) $\boxed{3 + 2\sqrt{5}}$ (**E**) None of these

49. Let $h(x) = \dfrac{16x^4 + 250x}{2x^3 + 5x^2 + 8x + 20}$ for $x \neq -\frac{5}{2}$. For $h(x)$ to be a continuous function, what value must be assigned to $h(x)$ when $x = -\frac{5}{2}$.

(**A**) $-\dfrac{2000}{41}$ (**B**) $\boxed{-\dfrac{1500}{41}}$ (**C**) $-\dfrac{500}{41}$ (**D**) $\dfrac{500}{9}$ (**E**) None of these

50. How many different triangles with integer side lengths can be created with two sides of length 14 and 19?

(**A**) 26 (**B**) $\boxed{27}$ (**C**) 31 (**D**) 32 (**E**) None of these