The 33rd Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 22, 2014 at Regional Testing Centers
Second Round: March 15, 2014 at The University of North Alabama

ALGEBRA II WITH TRIGONOMETRY EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptoanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

The following information is courtesy of the U.S. Bureau of Labor Statistics.

- The median salary of a Mathematician in 2012 was $101,360 per year.
- Over the next 10 years, the job opportunities for mathematicians are expected to grow by 23%!

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

1. If $2^{2014} - 2^{2013} - 2^{2012} + 2^{2011} = k \cdot 2^{2011}$, what is the value of k?

(**A**) 1 (**B**) 2 (**C**) $\boxed{3}$ (**D**) 4 (**E**) 5

2. How many real solutions are there to the equation $x^x = x$?

(**A**) 1 (**B**) $\boxed{2}$ (**C**) 3 (**D**) 4 (**E**) infinitely many

3. Which of the following is the correct ordering of the numbers $10^{1/10}, 2^{1/3}$ and $3^{1/5}$?

(**A**) $2^{1/3} < 3^{1/5} < 10^{1/10}$ (**B**) $2^{1/3} < 10^{1/10} < 3^{1/5}$ (**C**) $10^{1/10} < 3^{1/5} < 2^{1/3}$

(**D**) $3^{1/5} < 2^{1/3} < 10^{1/10}$ (**E**) $\boxed{3^{1/5} < 10^{1/10} < 2^{1/3}}$

4. The equation $x + \sqrt{x-2} = 4$ has

(**A**) 2 real roots (**B**) 1 real and 1 imaginary root

(**C**) 2 imaginary roots (**D**) no roots (**E**) $\boxed{\text{1 real root}}$

5. The largest number by which the expression $n^3 - n$ is divisible by for all possible integer values of n is:

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) $\boxed{6}$

6. If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$ then $f(3)$

(**A**) -5 (**B**) -2 (**C**) 1 (**D**) 3 (**E**) $\boxed{8}$

7. How many positive integers k are there with the property that $k!$ is not divisible by 100?

(**A**) 19 (**B**) 4 (**C**) 14 (**D**) $\boxed{9}$ (**E**) 15

8. A group of seven people are waiting for a taxi. The taxi only holds four people. In how many ways can they pick which four people get in the taxi?

(**A**) 210 (**B**) $\boxed{35}$ (**C**) 840 (**D**) 24 (**E**) 720

9. If $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$ for $k \neq 0$ then $r =$

(**A**) $-\log_2 5$ (**B**) $\log_5 2$ (**C**) $\log_{10} 5$ (**D**) $\boxed{\log_2 5}$ (**E**) $5/2$

10. A total of 28 handshakes were exchanged at the conclusion of a party. Assuming that each participant was equally polite towards all of the others, the number of people present was

(**A**) 14 (**B**) 28 (**C**) 56 (**D**) $\boxed{8}$ (**E**) 7

11. Evaluate $\sin 80^\circ \cos 65^\circ - \cos 80^\circ \sin 65^\circ$.

(**A**) $\dfrac{\sqrt{1-\sqrt{3}}}{2}$ (**B**) $\dfrac{\sqrt{2+\sqrt{3}}}{2}$ (**C**) $\boxed{\dfrac{\sqrt{2-\sqrt{3}}}{2}}$ (**D**) $\dfrac{\sqrt{1+\sqrt{3}}}{2}$ (**E**) $\dfrac{\sqrt{2-2\sqrt{3}}}{2}$

12. Simplify the expression $\left(\sqrt{2}+\sqrt{2}i\right)^{2014}$.

(**A**) -4^{2014} (**B**) -4^{1007} (**C**) $\boxed{-2^{2014}i}$ (**D**) 2^{2014} (**E**) $2^{1007}i$

13. If m and b are real numbers and $mb > 0$, then the line whose equation is $y = mx + b$ cannot contain the point

(**A**) $(0, 5)$ (**B**) $(0, -6)$ (**C**) $(10, 15)$ (**D**) $(6, -9)$ (**E**) $\boxed{(20, 0)}$

14. How many different prime numbers are factors of N if

$$\log_2(\log_3(\log_5(\log_7 N))) = 11$$

(**A**) $\boxed{1}$ (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 7

15. Which of the following angles satisfies $\cot\theta\cos\theta < 0$?

(**A**) 450° (**B**) $\boxed{\frac{27\pi}{5}}$ (**C**) $-\frac{\pi}{2}$ (**D**) $\frac{4\pi}{7}$ (**E**) -189°

16. Before Ashley started a 2-hour drive, her car's odometer reading was 27972, a palindrome (A *palindrome* is a number that reads the same backwards as forwards.) At her destination, the odometer reading was another palindrome. If Ashley never exceeded 75 mph, which of the following was her average speed?

(**A**) 50 mph (**B**) $\boxed{55 \text{ mph}}$ (**C**) 60 mph (**D**) 65 mph (**E**) 70 mph

17. The mean of 3 numbers is 10 more than the least of the numbers and 15 less than the greatest of the three. If the median is 5, then the sum of the 3 is

(**A**) 5 (**B**) 20 (**C**) 25 (**D**) $\boxed{30}$ (**E**) 36

18. The product of the roots of $x^2 - 4\sqrt{7} + \dfrac{28}{x^2}$ is

(**A**) -14 (**B**) $4\sqrt{7}$ (**C**) $-4\sqrt{7}$ (**D**) $2\sqrt{7}$ (**E**) $\boxed{-2\sqrt{7}}$

19. If $\dfrac{x}{y} = \dfrac{3}{4}$, then which of the following is an incorrect expression:

(**A**) $\dfrac{x+y}{y} = \dfrac{7}{4}$ (**B**) $\dfrac{y}{y-x} = \dfrac{4}{1}$ (**C**) $\dfrac{x+2y}{x} = \dfrac{11}{3}$

(**D**) $\boxed{\dfrac{x-y}{y} = \dfrac{1}{4}}$ (**E**) All of these are correct expressions.

20. Find the sum of all distinct three digit numbers that contain only the digits 1, 2, 3, 4, each at most once.

(**A**) 1250 (**B**) 1110 (**C**) $\boxed{6660}$ (**D**) 3230 (**E**) 8670

21. Evaluate $1 - 2 + 3 - 4 + \cdots - 98 + 99$.

(**A**) -50 (**B**) -49 (**C**) -0 (**D**) 49 (**E**) $\boxed{50}$

22. If $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find $\cos(3x)$.

(**A**) $\frac{14}{25}$ (**B**) $\frac{12}{5}$ (**C**) $-\frac{36}{125}$ (**D**) $\boxed{-\frac{44}{125}}$ (**E**) $\frac{4}{5}$

23. Evaluate the following:

$$\log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{98}{99}\right) + \log_{10}\left(\frac{99}{100}\right)$$

(**A**) 0 (**B**) 2 (**C**) $\boxed{-2}$ (**D**) -1 (**E**) 10

24. If there are six different pairs of socks in the dryer, what is the smallest number you could reach in and grab (without looking) to guarantee you have a matching pair?

(**A**) 2 (**B**) 3 (**C**) 6 (**D**) $\boxed{7}$ (**E**) 12

25. For exactly two real values of m, m_1 and m_2, the line $y = mx+3$ intersects the parabola $y = x^2+2x+7$ at exactly one point. Compute $m_1^2 + m_2^2$.

(**A**) 8 (**B**) $\boxed{40}$ (**C**) 25 (**D**) 24 (**E**) 68

26. If $9^{-x} = 7$, what is 27^{2x+1}?

(**A**) $\frac{27}{7\sqrt{7}}$ (**B**) $\frac{7\sqrt{7}}{27}$ (**C**) $\boxed{\frac{27}{343}}$ (**D**) $\frac{343}{27}$ (**E**) None of these

27. The following system has a unique solution (a, b) satisfying it.

$$\begin{cases} \frac{1}{x} + \frac{3}{y} &= 7 \\ \frac{4}{x} - \frac{2}{y} &= 1 \end{cases}$$

Find $\frac{1}{a} + \frac{1}{b}$.

(**A**) $\boxed{\frac{22}{7}}$ (**B**) $\frac{57}{7}$ (**C**) $\frac{62}{5}$ (**D**) $\frac{6}{5}$ (**E**) $\frac{78}{7}$

28. What is the remainder when the polynomial $x^4 + 4$ is divided by $x^2 - 5$?

(**A**) 25x+4 (**B**) $\boxed{29}$ (**C**) 0 (**D**) $x + 5$ (**E**) -21

29. A fair 10-sided die is rolled. What is the probability a prime number was rolled?

(**A**) $\frac{1}{2}$ (**B**) $\boxed{\frac{2}{5}}$ (**C**) $\frac{3}{10}$ (**D**) $\frac{3}{5}$ (**E**) $\frac{7}{10}$

30. Find the product of all solutions of the equation $\sin^2 x \sec x + 2\sin^2 x = \sec x + 2$ in the interval $[0, 2\pi)$.

Problem thrown out. The correct answer is $\dfrac{8\pi^2}{9}$.

(**A**) $\dfrac{2\pi^4}{3}$ (**B**) $\dfrac{35\pi^4}{48}$ (**C**) 0 (**D**) $\dfrac{4\pi^3}{9}$ (**E**) $\dfrac{35\pi^3}{72}$

31. How many integers satisfy $n^4 + 6n < 6n^3 + n^2$?

(**A**) 8 (**B**) $\boxed{4}$ (**C**) 0 (**D**) 5 (**E**) Infinitely many

32. There exist postive integers A, B and C, with no common factor greater than 1, such that

$$A \log_{200} 5 + B \log_{200} 2 = C$$

What is $A + B + C$?

(**A**) $\boxed{6}$ (**B**) 7 (**C**) 8 (**D**) 9 (**E**) 10

33. Which of the following graphs intersect the graph of $y = x^2 - 4$ the most times?

(**A**) $x^2 + y^2 = 1$ (**B**) $y = x + 5$ (**C**) $y = -x^2 + 4$ (**D**) $y = x^3$ (**E**) $\boxed{x^2 + y^2 = 9}$

34. One fun-size Snickers bar and two-fun size Milky Way bars contain 230 calories. Two fun-size Snickers bars and one fun-size Milky Way bar contain 235 calories. How many calories are there in a fun-size Milky Way?

(**A**) 64 (**B**) 70 (**C**) $\boxed{75}$ (**D**) 80 (**E**) 83

35. How many distinguishable permutations are there of the letters in the word CIRCLE?

(**A**) 60 (**B**) 720 (**C**) $\boxed{360}$ (**D**) 120 (**E**) 240

36. Define a function to be "nice" if $f(x + y) = f(x) + f(y)$ for any values of x and y in the domain. Which of the following functions is nice on its domain?

(**A**) $f(x) = \ln(x)$ (**B**) $f(x) = \sin(x)$ (**C**) $f(x) = \sqrt{x}$ (**D**) $\boxed{f(x) = 5x}$ (**E**) $y = |x|$

37. Find the product of the real solutions to the equation $(x^2 - x + 1)(x^2 - x + 2) = 12$.

(**A**) $\sqrt{21}$ (**B**) 0 (**C**) $\boxed{-2}$ (**D**) -3 (**E**) 6

38. Which of the following functions are one-to-one on their domain?

(**A**) $y = |x|$ (**B**) $y = \tan(x)$ (**C**) $y = \ln|x|$ (**D**) $\boxed{y = \dfrac{1}{x}}$ (**E**) $y = x^2 - 3$

39. A sequence is defined recursively as $a_1 = 2$ and $a_k = -2a_{k-1} + 3$ where $k \geq 2$. What is the median of the first six terms of the sequence?

(**A**) -1 (**B**) -2.5 (**C**) $\boxed{0.5}$ (**D**) 5 (**E**) 2

40. If $\sin t = a$, $\cos t = b$ and $\tan t = c$, then the expression

$$\sin(-t + 4\pi) + 3\cos(\frac{\pi}{2} - t) - \tan(t - 3\pi)$$

can be written as

(**A**) $a - 3b - c$ (**B**) $-a + 3b + c$ (**C**) $-a + 3b + \frac{c}{2}$ (**D**) $\boxed{2a - c}$ (**E**) $4a + c$

41. In a recent survey, 50 students are asked about which type of gaming system they have. Of those surveyed, 33 say that they have a PS4 and 29 say that they have an Xbox One. If 19 of the students have both, how many have neither a PS4 nor an Xbox One?

(**A**) 12 (**B**) 4 (**C**) $\boxed{7}$ (**D**) 17 (**E**) 2

42. Find the largest real value of x such that the reciprocal of $x + 1$ is $x - 1$.

(**A**) $\boxed{\sqrt{2}}$ (**B**) $\sqrt{5}$ (**C**) -2 (**D**) $\sqrt{6}$ (**E**) 0

43. If $xy = 2$, $yz = 3$ and $xz = 5$, what is the value of $x^2 + y^2 + z^2$?

(**A**) $\boxed{\frac{361}{30}}$ (**B**) 20 (**C**) $\frac{6}{5}$ (**D**) $\frac{65}{6}$ (**E**) 32

44. The solution sets of the polynomial equations $f(x) = 0$ and $g(x) = 0$, are, respectively, $\{-2, 0, 3, 7\}$ and $\{1, 3, 4\}$. How many different numbers are in the solution set of the polynomial equation $f(x)g(x) = 0$?

(**A**) 4 (**B**) 5 (**C**) $\boxed{6}$ (**D**) 7 (**E**) none of the above

45. The equation $4x^2 - y^2 = 11$ has exactly one integer pair solution (x, y) with both $x, y > 0$. Find x.

(**A**) 5 (**B**) -5 (**C**) -3 (**D**) $\boxed{3}$ (**E**) 1

46. How many solutions are there to the equation $\left|2x^2 - x - 1\right| = x$?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) 4

47. Define a function as follows: for $x > 0$, $f(x) = f(x - 2)$ and for $x \leq 0$, $f(x) = |x|$. Find the value of $f(2.7) - f(5)$.

(**A**) 2.3 (**B**) $-.7$ (**C**) -2.3 (**D**) $\boxed{.3}$ (**E**) 1.7

48. One of the zeros of the function $f(x) = x^4 + 5x^3 + 10x^2 + 20x + 24$ is $2i$, where $i = \sqrt{-1}$. What is the sum of the the real zeros of $f(x)$?

(**A**) $\boxed{-5}$ (**B**) 4 (**C**) 0 (**D**) 5 (**E**) -4

49. Consider the ellipse given by the equation $9x^2 + 4y^2 - 18x + 16y - 11 = 0$. Let c be the length of the major axis and d be the length of the minor axis. Find cd.

(**A**) 36 (**B**) $\boxed{24}$ (**C**) 6 (**D**) 16 (**E**) 18

50. With a rational denominator, the expression $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ is equivalent to

(**A**) $\boxed{\dfrac{3+\sqrt{6}+\sqrt{15}}{6}}$ (**B**) $\dfrac{\sqrt{6}-2+\sqrt{10}}{6}$

(**C**) $\dfrac{2+\sqrt{6}+\sqrt{10}}{10}$ (**D**) $\dfrac{2+\sqrt{6}-\sqrt{10}}{6}$ (**E**) None of these