Solutions for 2015 Algebra II with Trigonometry Exam

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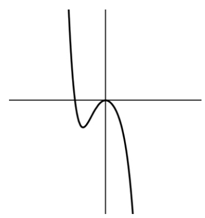
1. For $f(x) = 6x - 4(2x-1)^2 + 5$, find $f(2)$.

   Solution: The notation $f(2)$ tells us to evaluate the function $f(x)$ at $x = 2$. Thus we have

   $$\begin{aligned} f(2) &= 6(2) - 4(3)^2 + 5 \\ &= 12 - 4 \cdot 9 + 5 \\ &= -24 + 5 \\ &= -19 \end{aligned}$$

   Therefore the answer is $\boxed{-19}$.

2. Which of the following could be the leading term of the polynomial whose graph is as pictured?

   Solution: The end behavior of the graph indicates it has an negative, odd degree leading term. Though the first thing that likely comes to mind is cubic, any odd power could yield this graph. Thus the best answer from the available choices is $\boxed{-2x^5}$.

3. What is the largest solution of the equation $2x^3 - 5x^2 = 8x - 20$?

   Solution: To solve this equation, we can move everything to the left side, factor by grouping, and then use the zero-product property to solve.

   $$\begin{aligned} 2x^3 - 5x^2 - 8x + 20 &= 0 \\ x^2(2x-5) - 4(2x-5) &= 0 \\ (x^2-4)(2x-5) &= 0 \\ (x-2)(x+2)(2x-5) &= 0 \end{aligned}$$

   Thus the solutions are $x = 2, -2, 5/2$ and so the largest solution is $5/2$. So the correct answer is $\boxed{\text{None of These}}$.

   Note: Because of the presence of the None of These answer choice, it is not enough to simply check the available answer choices and choose the largest one that works.

4. Which of the following functions is/are equal to $f(x) = 4^x$?

   Solution: It helps to start with 4^x and think about exponent laws. We can write 4 as 2^2 and so $4^x = (2^2)^x$. When we have a power raised to another power, we multiply the powers. Thus $4^x = (2^2)^x = 2^{2x}$. So we know that whatever the answer is, it must include III. Again by exponent laws, $(2^x)^2 = 2^{2x}$, and so we also know that II and III are equivalent. To see that I is not equivalent to these, you'll need to remember that when you have nested exponents like this (one value raised to another, raised to another, etc.), the order of operations works from the top down. That is, for example, if you had 2^{2^3}, you would first perform the 2^3 to get 8 and then perform 2^8. Thus by simple example (using $x = 3$) you can see that I would yield 2^8, while II and III both give 2^6. Thus the answer is $\boxed{\text{II and III only}}$.

5. The number of values satisfying the equation $\dfrac{2x^2 - 10x}{x^2 - 5x} = x - 3$ is:

   Solution: To solve this rational equation, multiply both sides by $x^2 - 5x$ and then solve the resulting cubic equation.

$$\begin{aligned} 2x^2 - 10x &= (x-3)(x^2 - 5x) \\ 2x^2 - 10x &= x^3 - 8x^2 + 15x \\ x^3 - 10x^2 + 25x &= 0 \\ x(x^2 - 10x + 25) &= 0 \\ x(x-5)^2 &= 0 \end{aligned}$$

   Thus we get $x = 0$ and $x = 5$ as two possible solutions. However, our original equation had domain restrictions. Checking both answers shows that both make the denominator of the rational expression on the left hand side of the equation 0, and thus are not solutions to the equation. Therefore the equation as $\boxed{0}$ solutions.

6. The difference quotient of a function $f(x)$ is the quotient $\dfrac{f(x+h) - f(x)}{h}, h \neq 0$. Find the difference quotient of $f(x) = \frac{3}{x}$.

   Solution: Recall that $f(x+h)$ means that we evaluate the function f at $x+h$. Thus plugging in $x+h$ where we see x, we get that $f(x+h) = \frac{3}{x+h}$. Now we can put this together in the difference quotient formula.

$$\frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \frac{\frac{3x}{x(x+h)} - \frac{3(x+h)}{x(x+h)}}{h} = \frac{\frac{3x-(3x+3h)}{x(x+h)}}{h} = \frac{\frac{3x-3x-3h}{x(x+h)}}{h}$$

   Combining denominators, and simplifying further, we get

$$\frac{\frac{3x-3x-3h}{x(x+h)}}{h} = \frac{-3h}{x(x+h)h} = \frac{-3}{x(x+h)}$$

   Therefore the answer is $\boxed{\frac{-3}{x(x+h)}}$.

7. In a warehouse, a stack of 6 foam mattress toppers are piled up. Each mattress topper is originally 3 inches thick. Each is compressed by a third each time an additional mattress topper is piled on top. Rounded to the nearest inch, which of the following is the total height of the pile?

   Solution: We are finding the sum of heights of mattresses, where each is decreasing in size by 1/3 each iteration. Thus the very top mattress is the original height of 3, the second is compressed to 2/3 of 3, or 2, the third is compressed to 2/3 of 2, or 4/3, etc. Using either a partial geometric sum formula, or simply adding together the approximations of the six terms, you get $3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} \approx 8$. Thus the height is approximately $\boxed{\text{8 inches}}$.

   Alternatively, you can make the finite geometric series

$$\sum_{k=0}^{5} 3\left(\frac{2}{3}\right)^k$$

   and use a partial sum formula to arrive at the answer.

8. Find the sum of the solutions to the equation $(3x - 10)(x + 1) = 10$

   Solution: Even though the left hand side is factored already, it is not set equal to 0. Thus we'll have to multiply out the left hand side, move the 10 over and then solve the resulting quadratic using the zero product property.

$$\begin{aligned}(3x-10)(x+1) &= 10\\ 3x^2-7x-10 &= 10\\ 3x^2-7x-20 &= 0\\ (3x+5)(x-4) &= 0\end{aligned}$$

Therefore there are two solutions $x=-\frac{5}{3}$ and $x=4$. Thus the solution is $-\frac{5}{3}+4=\boxed{\frac{7}{3}}$

9. The student council is made up of four sophomores, two juniors and three seniors. A yearbook photographer would like to line up all members of the student council in a line for a picture. How many different pictures are possible if students in the same grade stand beside each other?

   Solution: Since the students of the same grade need to stand together, we need to arrange four things: Sophomores, Juniors, Seniors and then also the grde levels themselves. That is, there is 4!2!3! ways to arrange the students within their groups in the order sophomores, juniors then seniors, but there are 3! ways to arrange each of the groups (seniors first, then juniors, sophomores; seniors first, then sophomores, juniors; etc.). Thus the final answer is $\boxed{4!2!3!3!}$.

10. For $f(x)$ defined piecewise below, evaluate $f(f(f(f(-3))))$.

$$f(x)=\begin{cases}|x| & \text{if} \quad x<-2\\ -5 & \text{if} \quad -2\le x\le 3\\ \frac{1}{x} & \text{if} \quad x>3\end{cases}$$

   Solution: To evaluate $f(f(f(f(-3))))$, we work our way from the inside outwards. Begin with $f(-3)$. Since $-3<-2$, $f(-3)=|-3|=3$. We now have $f(f(f(3)))$. Since $x=3$ falls into the middle condition of the piecewise, then $f(3)=-5$. Thus we now have $f(f(-5))$. Again as $-5<-2$, $f(-5)=|-5|=5$. Finally, we have simplified down to $f(5)$. As $5>3$, $f(5)=\frac{1}{5}$. Therefore $f(f(f(f(-3))))=\boxed{\frac{1}{5}}$.

11. Find the horizontal asymptote of the graph of the function $f(x)=\dfrac{-6x^2+5x}{4x+5}+\dfrac{12x^3+7x+1}{8x^2+6x-5}$.

   Solution: In order to find the horizontal asymptote, you need the combine the terms into a single rational function. Notice that $8x^2+6x-5=(4x+5)(2x-1)$. Thus to get a common denominator, we need only multiply the first term by $\frac{2x-1}{2x-1}$. This gives

$$\frac{-6x^2+5x}{4x+5}\left(\frac{2x-1}{2x-1}\right)+\frac{12x^3+7x+1}{8x^2+6x-5}=\frac{-12x^3+16x^2-5x}{8x^2+6x-5}+\frac{12x^3+7x+1}{8x^2+6x-5}=\frac{16x^2+2x+1}{8x^2+6x-5}$$

   Now that we have a single rational function, the horizontal asymptote is $y=a$, where a is the ratio of the coefficient of the leading term of the numerator to the coefficient of the leading term of the denominator (since the degrees of the leading terms are the same): $\frac{16}{8}=2$. Thus $\boxed{y=2}$ is the horizontal asymptote.

12. If $x+\dfrac{1}{x}=3$, then the value of $x^3+\dfrac{1}{x^3}$ is

   Solution: Notice that since we are wanting the value of $x^3+\dfrac{1}{x^3}$, we should cube $x+\dfrac{1}{x}$. Expanding $\left(x+\frac{1}{x}\right)^3$ we get the expression: $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$. Cubing the right hand side as well gives the equation

$$x^3+3x+\frac{3}{x}+\frac{1}{x^3}=27$$

   Now, rearranging, we get

$$x^3+\frac{1}{x^3}+3x+\frac{3}{x}=27$$

   As $3x+\frac{3}{x}=3\left(x+\frac{1}{x}\right)$, and by assumption, $x+\dfrac{1}{x}=3$, we have $3x+\frac{3}{x}==3\times 3=9$. Substituting this in, we get

$$x^3+\frac{1}{x^3}+9=27$$

and so $x^3 + \frac{1}{x^3} = \boxed{18}$.

13. Find the slope of the line perpendicular to the line connecting the two points $(\frac{1}{4}, -\frac{1}{3})$ and $(-\frac{1}{6}, 1)$.

    Solution: The formula for the slope of a line between two points is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Thus, the slope is:

    $$m = \frac{1 - \frac{-1}{3}}{-\frac{1}{6} - \frac{1}{4}} = \frac{\frac{4}{3}}{-\frac{5}{12}} = \frac{4}{3} \cdot \frac{-12}{5} = -\frac{16}{5}$$

    Now, the question asks for the slope of a line perpendicular to the line connecting the points. The slope of a perpendicular line is the negative reciprocal of the slope of the original line, and so the slope of the perpendicular line is $\boxed{\frac{5}{16}}$.

14. In the senior class at a particular high school, 45 students are taking calculus, 52 students are taking physics, and 21 students are taking both calculus and physics. If there are 200 people in the senior class, what is the probability that a randomly selected student is taking calculus or physics?

    Solution: To solve this problem, we can either fill out a Venn Diagram, or use the Union Rule. The union rule states that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$, for events E and F. Since we are looking for the probability of taking calculus or physics, we are solving for $P(E \cup F)$ in the equation. Now, $P(\text{taking calculus}) = 45/200$, $P(\text{taking physics}) = 52/200$ and $P(\text{taking calculus and physics}) = 21/200$. Thus

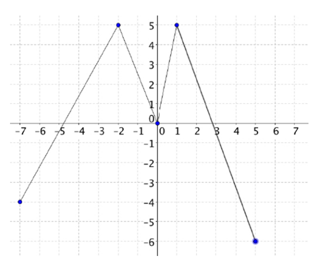
    $$\begin{aligned} P(\text{taking calculus or physics}) &= P(\text{taking calculus}) + P(\text{taking physics}) + P(\text{taking calc and physics}) \\ &= 45/200 + 52/200 - 21/200 \\ &= 76/200 \end{aligned}$$

    Thus $P(\text{taking calculus or physics}) = 76/200$ or $\boxed{0.38}$.

15. To which of the following expressions is $\sqrt[8]{x^8} + \sqrt[7]{x^7}$ equal to for all real negative values of x?

    Solution: Recall that $\sqrt[n]{x^n} = |x|$ for even n and $\sqrt[n]{x^n} = x$ for odd n. Therefore for all values of x, $\sqrt[8]{x^8} = |x|$ and $\sqrt[7]{x^7} = x$. But since $x < 0$ in this question, we know that $|x| = -x$. Thus $\sqrt[8]{x^8} + \sqrt[7]{x^7} = |x| + x = -x + x = 0$. Therefore the correct answer, for negative values of x, is $\boxed{0}$.

16. The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 5$ have?

    Solution: This problem will involve solving two equations: for which z are $f(z) = 5$ and then for which values of x do $f(x) = z$. To begin with, we can see that $z = -2$ and $z = 1$ both have $f(z) = 5$. For which x do $f(x) = -2$ or $f(x) = 1$? For $f(x) = -2$, we have two values of x. It's hard to tell exactly what they are, but approximately, $x = -6$ and $x = 3.5$ both have $f(x) = -2$ and so $f(f(x)) = f(-2) = 5$. For $f(x) = 1$, we can see there are four total values. Their approximate values are $x = -4.25, x = -.5, x = .25$ and $x = 2.5$, all of which have $f(x) = 1$ and so $f(f(x)) = f(1) = 5$. Thus there are a total of $\boxed{6}$ values for which $f(f(x)) = 5$.

17. Exactly how many solutions are there to the equation $|x - 2| + |x - 3| = 1$?

    Solution: Recall that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. Applying similar logic, we can see that $|x - 2| = x - 2$ when $x \geq 2$ and $|x - 2| = -(x - 2) = 2 - x$ when $x < 2$ and $|x - 3| = x - 3$ when $x \geq 3$ and $|x - 3| = -(x - 3) = 3 - x$ when $x < 3$. This breaks the real numbers into three intervals,

$(-\infty, 2)$, $[2, 3)$ and $[3, \infty)$. We will solve the equation in each of these three intervals, and that will give us all solutions. For $(-\infty, 2)$, we know that both $|x-2| = 2-x$ and $|x-3| = 3-x$. Thus the equation simplifies to $2-x+3-x = 1$. Further work with this gives $5-2x = 1$, which gives a single answer of $x = 2$. [Technically this is in the next interval, but we can count it either place. The choice of where to include 2 was arbitrary]

Now for the interval $[2, 3)$, we know that $|x-2| = x-2$ and $|x-3| = 3-x$, since all values on $[2, 3)$ are less than $x = 3$. Thus this gives the equation $x-2+3-x = 1$. However, this simplifies to the statement $1 = 1$, which is always true. Thus all values in the interval $[2, 3)$ satisfy the equation. Therefore there are infinitely many solutions to this equation and so the answer is $\boxed{\text{None of These}}$.

Note: We could finish the problem of finding all solutions by solving the equation on $[3, \infty)$, but as the problem was only asking for the number, we do not need to.

18. If $\frac{2}{5}$ and $\frac{14}{15}$ are the first and fifth terms of an arithmetic sequence, what is the sum of the second, third and fourth terms of the sequence?

Solution: Since we are told that $\frac{14}{15}$ is the fifth term of the sequence, we can set up an equation to solve. The five terms of the sequence will be $\frac{2}{5}$, $\frac{2}{5}+x$, $\frac{2}{5}+2x$, $\frac{2}{5}+3x$, $\frac{2}{5}+4x = \frac{14}{15}$. Solving for x in this equation, we get $x = \frac{2}{15}$. Thus the second, third and fourth terms are $\frac{8}{15}$, $\frac{10}{15}$ and $\frac{12}{15}$. Their sum is then $\frac{30}{15}$ or $\boxed{2}$.

19. Suppose that $f(x) = x^2 - 6x$, $g(x) = x+3$ and $h(x) = \sqrt{x}$. Find $K(x)$, where $K(x) = (f \circ (g \circ h))(x)$.

Solution: To find $K(x)$ we will follow the order of composition indicated in the problem. First, we compose $(g \circ h)(x)$.

$$(g \circ h)(x) = g(h(x)) = g(\sqrt{x}) = \sqrt{x} + 3$$

Then,

$$f(\sqrt{x}+3) = (\sqrt{x}+3)^2 - 6(\sqrt{x}+3) = x + 6\sqrt{x} + 9 - 6\sqrt{x} - 18$$

Simplifying, we get $K(x) = \boxed{x-9}$.

20. Let f be a function satisfying $f(xy) = \dfrac{f(x)}{y}$ for all positive real numbers x and y. If $f(500) = 3$, what is the value of $f(600)$?

Solution: To find $f(600)$, knowing only the value of $f(500)$ we must write 600 in terms of 500. We can do this as: $600 = 500 \cdot \frac{6}{5}$. Then

$$f(600) = f(500 \cdot \frac{6}{5}) = \frac{f(500)}{6/5} = f(500) \cdot \frac{5}{6} = 3 \cdot \frac{5}{6} = \frac{5}{2}$$

Thus the value of $f(600)$ is $\boxed{\frac{5}{2}}$.

21. The graph of the function $y = \dfrac{x^2 - 5x + 6}{2x^2 + 3x - 4}$ has two vertical asymptotes $x = a$ and $x = b$. Find ab.

Solution: Vertical asymptotes occur when the denominator is zero, but the numerator is non-zero. Using the quadratic formula, we find the two roots of the polynomial in the denominator are $x = \frac{-3 \pm \sqrt{41}}{4}$. The question asks for the product, and so $\frac{-3+\sqrt{41}}{4} \cdot \frac{-3-\sqrt{41}}{4} = \frac{9-41}{16} = \boxed{-2}$.

22. A line with slope 3 intersects a line with slope $-\frac{1}{2}$ at the point $(14, 15)$. What is the distance between the x-intercepts of these lines?

Solution: To find the distance between x-intercepts, we first need to find the equations of the two lines. The first line has slope $m_1 = 3$ going through point $(14, 15)$. Thus

$$\begin{aligned} y &= mx + b \\ 15 &= 3(14) + b \\ 15 &= 42 + b \\ b &= -27 \end{aligned}$$

So the equation of the first line is $y = 3x - 27$. Similarly, we can find that the equation of the second line is $y = -\frac{1}{2}x + 22$. Now, to find the x-intercepts, we set $y = 0$ and solve.

$$\begin{aligned} 0 &= 3x - 27 \\ 27 &= 3x \\ 9 &= x \end{aligned}$$

Thus the x-intercept of the line $y = 3x - 27$ is $(9, 0)$. Similarly, we can find the x-intercept of the line $y = -\frac{1}{2}x + 22$ is $(44, 0)$. Finally, the distance between these points is

$$d = \sqrt{(44-9)^2 + (0-0^2)} = \sqrt{35^2} = 35$$

So the answer is $\boxed{35}$.

23. Six houses in a row are each to be painted with one of the colors Red, Blue, Green and Yellow. In how many different ways can the houses be painted so that no two adjacent houses are the same color?

Solution: To solve this problem, we will consider for each house how many color options we can use. Then, we'll use the multiplication principle. For the first house, we can use any of the four colors. For the second house, the only house that has been painted is the first, so we only have to avoid that color. Thus the second house has 3 color options. For the third house, only one of its adjacent neighbors has been painted, and so again we only have one color we must avoid. Thus the third house also has 3 color options. Similarly, all of the remaining houses have 3 color options, as they only need to avoid the color of their painted neighbor to the left. Thus we have

4	3	3	3	3	3
House 1	House 2	House 3	House 4	House 5	House 6

giving us $4 \cdot 3^5 = \boxed{972}$ ways to paint the houses.

24. A fixed point of a function f is a value c in the domain of f such that $f(c) = c$. Find the number of fixed points of the function $f(x) = \dfrac{1}{x+2}$.

Solution: To find the number of fixed points of $f(x)$, we just need to find all c such that $f(c) = c$. This comes down to solving the equation $\frac{1}{c+2} = c$. Multiplying both sides by $c+2$ we get $1 = c(c+2) = c^2+2c$. Rearranging that, we get the $c^2 + 2c - 1 = 0$. This expression does not easily factor, but either the quadratic formula or completeting the square shows that this equation has two solutions: $c = -1 + \sqrt{2}$ and $c = -1 - \sqrt{2}$. Thus there are $\boxed{2}$ fixed points for $f(x)$.

25. Which of the following best describes the graph of the equation $9x^2 - 54x = 4y^2 + 16y - 29$?

Solution: The easiest way to tell the shape of such an equation is to get it into standard form. Moving the variables to the left hand side and rearranging gives us $9x^2 - 54x - 4y^2 - 16y = -29$. We can complete the square to get it into standard form:

$$\begin{aligned} 9(x^2 - 6x) - 4(y^2 + 4y) &= -29 \\ 9(x^2 - 6x + 9) - 4(y^2 + 4y + 4) &= -29 + 81 - 16 \\ 9(x-3)^2 - 4(y+2)^2 &= 36 \\ \tfrac{(x-3)^2}{4} - \tfrac{(y+2)^2}{9} &= 1 \end{aligned}$$

This is the standard form of a $\boxed{\text{Hyperbola}}$.

26. The equation $2x^{5/6} + 2x^{1/2} = 5x^{2/3}$ has two nonzero real solutions, a and b. Assuming $a < b$, find $\frac{1}{a} + b$.

Solution: To begin with, it will help to rewrite the problem with everything moved to one side, and to rewrite the exponents with a common denominator. Doing this, we get the equation

$$2x^{5/6} - 5x^{4/6} + 2x^{3/6} = 0$$

Now, you can see that each of the three terms share an $x^{3/6}$. We will factor it out:

$$x^{3/6}(2x^{2/6} - 5x^{1/6} + 2) = 0$$

The factor $x^{3/6}$ will result in a solution of 0, but since the problem asked for nonzero solutions, we can ignore this factor. This leaves us with a quadratic-like equation of

$$2x^{2/6} - 5x^{1/6} + 2 = 0$$

Let $z = x^{1/6}$. Then we can write our equation as

$$2z^2 - 5z + 2 = 0$$

This factors as $(2z-1)(z-2) = 0$, which gives solutions of $z = \frac{1}{2}$ and $z = 2$. Finally, after we substitute x back in, we have to solve the equations $x^{1/6} = \frac{1}{2}$ and $x^{1/6} = 2$. Raising both sides to the sixth power gives that $x = \frac{1}{2^6} = \frac{1}{64}$ and $x = 2^6 = 64$. Lastly, the problem wants the answer in the form $\frac{1}{a} + b$, where a is the smaller of the two solutions, which gives us $64 + 64 = \boxed{128}$.

27. Suppose $f(1) = 3$ and for all integers $n > 1$, $f(n) = f(n-1) + 1$ if n is even and $f(n) = 2f(n-1)$ if n is odd. Find $f(5)$.

    Solution: To find $f(5)$, we iterate until we can use $f(1) = 3$ to get a final answer.

$f(5) =$	$2f(5-1)$	Since 5 is odd, we use the rule for odd numbers
$=$	$2f(4)$	
$=$	$2\left(f(4-1)+1\right)$	Since 4 is even, we use the rule for even numbers
$=$	$2\left(f(3)+1\right)$	
$=$	$2\left(2f(3-1)+1\right)$	Replace $f(3)$ with $2f(3-1)$
$=$	$2\left(2f(2)+1\right)$	
$=$	$2\left(2[f(2-1)+1]+1\right)$	
$=$	$2\left(2[f(1)+1]+1\right)$	
$=$	$2\left(2[3+1]+1\right)$	Replace $f(1)$ with 3, the given
$=$	$2(2[4]+1)$	
$=$	$2(9)$	
$=$	18	

    Thus $f(5) = \boxed{18}$.

28. Find the remainder when the polynomial $f(x) = 2x^4 + 5x^3 - 3x - 8$ is divided by the polynomial $x + 2$.

    Solution: There are several ways to do this problem: Long division, synthetic division, or the Remainder Theorem. The Remainder Theorem states that the remainder when you divide a polynomial $P(x)$ by a linear factor $x - a$ is just $P(a)$, the polynomial evaluated at a. In our case, the $a = -2$ [as $x + 2 = x - (-2)$], and so the remainder we are finding is simply $f(-2) = 2(-2)^4 + 5(-2)^3 - 3(-2) - 8 = -10$. Therefore the answer was $\boxed{\text{None of These}}$.

29. On an island, 99% of the population are natives. Some natives emigrate so that only 98% of the population are natives. If the initial population of the island was 1000, how many natives emigrated?

    Solution: If the initial population is 1000, and 99% of those are natives, then the breakdown at the beginning is $99\% \times 1000 = 990$ natives and $1000 - 990 = 10$ non-natives. Some natives then emigrated. However, no non-natives emigrated. Thus the 10 non-natives, all of whom are still present, are now 2% of the population (since 98% is now native). We can then set up the equation:

$$2\% \times p = 10$$

    where p is the new total population. Solving this equation gives $p = 500$. Therefore the new population of the island is 500, and thus the population of the island decreased by 500 people. However, all who emigrated were native, and thus $\boxed{500}$ natives emigrated.

30. Find the inverse function $f^{-1}(x)$ of the function $f(x) = \dfrac{x+2}{x+3}$.

Solution: One method of finding an inverse function is to swap x and y in the original function, and then re-solve for y. Thus we have

$$\begin{aligned} x &= \tfrac{y+2}{y+3} && \text{Swap x and y} \\ x(y+3) &= y+2 && \text{Multiply both sides by } y+3 \\ xy+3x &= y+2 && \text{Distribute} \\ xy-y &= -3x+2 && \text{Rearrange} \\ y(x-1) &= 2-3x && \text{Factor out a } y \\ y &= \tfrac{2-3x}{x-1} && \text{Divide both sides by } x-1 \end{aligned}$$

Thus the inverse is $\boxed{f^{-1}(x) = \frac{2-3x}{x-1}}$.

Caution: $f^{-1}(x)$ is the function which composes with $f(x)$ to yield the identity. That is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. In general, it is **not** the same as $\frac{1}{f(x)}$, the reciprocal of the function.

31. Find the product of all real numbers k for which the function $f(x) = kx^2 + x + k$ touches, but does not cross, the x-axis.

Solution: The are multiple ways to approach this problem. One is to realize that $f(x)$ is a parabola. If a parabola is to touch, but not cross the x-axis at a point, then it must have a double root at that point. That is, it must be able to be written as a square. If you are to square a linear term $ax+b$ and get kx^2 and k as two of the terms, we must have it factor as $kx^2+x+k = (\sqrt{k}x \pm \sqrt{k})^2$. Expanding the square gives two equations: $1x = 2k$ and $1x = -2k$. Solving these equations gives $k = \frac{1}{2}$ and $k = -\frac{1}{2}$ and so the product is $\boxed{-\frac{1}{4}}$

Another approach: First, we factor $f(x)$ into standard form. As it touches but does not cross the x-axis, it will factor as $f(x) = a(x-h)^2$. Expanding this we get $f(x) = ax^2 - 2ahx + ah^2$. However, we also know that $f(x) = kx^2 + x + k$. Equating coefficients of terms of the same degree, we get three equations: $a = k$, $-2ah = 1$ and $ah^2 = k$. Combining the first and third equations, we can conlcue that $h^2 = 1$ and so $h = \pm 1$. We can then substitute these into the equation $-2ah = 1$ and conclude that $a = \pm\frac{1}{2}$. Thus the product is $-\frac{1}{4}$

32. The graphs of equations $9x^2 + y^2 = 9$ and $3x + 2y = 6$ intersect at points (a, b) and (c, d). Find $a + b + c + d$.

Solution: We need to solve the non-linear system:

$$\begin{cases} 9x^2 + y^2 = 9 \\ 3x + 2y = 6 \end{cases}$$

To do this, we can solve for y in the linear equation, and then substitute it into the other equation. (You could also solve for x, but then you would have to substitute that into $9x^2$, which requires a little more work that substituting into y^2.) Solving for y gives $y = 3 - \frac{3}{2}x$.

$$\begin{aligned} 9x^2 + y^2 &= 9 \\ 9x^2 + \left(3 - \tfrac{3}{2}x\right)^2 &= 9 \\ 9x^2 + 9 - 9x + \tfrac{9}{4}x^2 &= 9 \\ x^2 + 1 - x + \tfrac{1}{4}x^2 &= 1 \\ \tfrac{5}{4}x^2 - x + 1 &= 1 \\ \tfrac{5}{4}x^2 - x &= 0 \\ x\left(\tfrac{5}{4}x - 1\right) &= 0 \end{aligned}$$

Thus $x = 0$ and $x = \frac{4}{5}$. To find the y values, we can substitute back into either equation. We'll use the one that is already solved for y:

$$y = 3 - \tfrac{3}{2}(0) = 3$$
$$y = 3 - \tfrac{3}{2}\left(\tfrac{4}{5}\right) = \tfrac{9}{5}$$

Thus the answer is $0 + 3 + \frac{4}{5} + \frac{9}{5} = \boxed{\frac{28}{5}}$

33. Let $f(x)$ be a function such that $f(x) + f\left(\frac{1}{1-x}\right) = x$ for all x not equal to 0 or 1. What is the value of $f(2)$?

Solution: To find $f(2)$, plug $x = 2$ into the given equation to get $f(2) + f\left(\frac{1}{1-2}\right) = 2$, which simplifies to

$$f(2) + f(-1) = 2$$

However, we do not know the value of $f(-1)$. To find it, we can plug $x = -1$ into the given equation to get $f(-1) + f\left(\frac{1}{1-(-1)}\right) = -1$, which simplifies to

$$f(-1) + f\left(\frac{1}{2}\right) = -1$$

This again presents a problem that we do not know the value of $f\left(\frac{1}{2}\right)$, and so we can try again to plug $x = \frac{1}{2}$ into the given equation. This gives us $f\left(\frac{1}{2}\right) + f\left(\frac{1}{1-1/2}\right) = \frac{1}{2}$. Simplifying, we get

$$f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2}$$

This gives us a system of three equations in three unknowns:

$$\begin{aligned} f(2) + f(-1) &= 2 \\ f(-1) + f\left(\tfrac{1}{2}\right) &= -1 \\ f\left(\tfrac{1}{2}\right) + f(2) &= \tfrac{1}{2} \end{aligned}$$

If you multiply the second equation through by -1 and then add all three equations, you get

$$2f(2) = \frac{7}{2}$$

and so $f(2) = \boxed{\frac{7}{4}}$.

34. The following transformations are applied (in the given order) to the graph of $y = |x|$.

    I. Reflection about the x-axis
    II. Horizontal shift left 3 units
    III. Vertical shift up 1 unit

Determine the equation of the graph produced as a result of applying these transformations.

Solution: To first reflect about the x-axis, multiply by -1, giving the equation $y = -|x|$. To shift left by 3 units, we replace x by $x + 3$, which gives $y = -|x + 3|$. Finally to shift up by 1 unit, we add 1, which gives $y = -|x + 3| + 1$. Thus the answer was $\boxed{\text{None of These}}$.

35. Find the sum of all solutions (both real and complex) to the equation $ix^2 + 7x - 12i = 0$, where $i = \sqrt{-1}$.

Solution: Despite the complex coefficients, you can solve this equation just like any other quadratic. Let's solve by factoring. If you like, you can first factor out an i to get

$$i(x^2 - 7ix - 12) = 0$$

However, we can also factor as $(ix+4)(x-3i) = 0$. Solving these two linear equations gives $x = \frac{-4}{i} = 4i$ and $x = 3i$. Thus the sum of all solutions is $\boxed{7i}$.

36. Define an operation # on pairs of real numbers as

$$(x_1, y_1)\#(x_2, y_2) = (x_1^2 x_2^2, y_1 y_2).$$

Which of the following could **not** equal $(x_1, y_1)\#(x_2, y_2)$ for any real numbers x_1, y_1, x_2, y_2?

Solution: Notice first that the first term in $(x_1^2 x_2^2, y_1 y_2)$ is a perfect square: $(x_1 x_2)^2$. Since this operation is defined only on real numbers, this must be positive. Thus the pair $\boxed{(-1, 0)}$ cannot be $(x_1, y_1)\#(x_2, y_2)$ for any real numbers numbers x_1, y_1, x_2, y_2.

37. The straight lines $-ax + \frac{1}{2}y = 1$ and $(a+1)x + y = 1$ are parallel to each other. Find the value of the constant a.

Solution: Since we are told the lines are parallel, we know they have the same slope. First, we need to find the slopes. Then we can set them equal to each other and solve the resulting equation. Solving for y in both, we see that the first line can be written as $y = 2ax+2$ and the second line as $y = -(a+1)x+1$. Thus the slopes are $2a$ and $-(a+1)$ respectively. Setting these equal to each other we get the equation $2a = -(a+1) = -a - 1$ which gives $a = \boxed{-\frac{1}{3}}$.

38. A box contains 1 white, 3 purple, and 2 gold balls. A second box contains 2 purple and 2 gold balls. One ball is selected at random from each box. What is the probability they are the same color?

Solution: There are two ways that both balls could be the same color: either both are purple or both are gold. Notice that both can't be white, since the second box does not contain white. The probability both are purple is the probability that you drew a purple from the first and a purple from the second: $\frac{3}{6} \times \frac{2}{4} = \frac{1}{4}$. The probability both are gold is the probability that you drew a gold from the first and a gold from the second: $\frac{2}{6} \times \frac{2}{4} = \frac{1}{6}$. To get the final answer, we add these two answers together, since we want the probability of one or the other. Thus the final answer is $\frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$.

39. Let a and d be real numbers such that $ax^3 + 6x^2 + 9x + d = (f(x))^3$ for some polynomial function $f(x)$. What is the value of ad?

Solution: Since $f(x)$ is a polynomial, which when cubed gives a cubic, we know that $f(x)$ is linear. Thus it can be written as $f(x) = bx + c$. Now, we will cube $f(x)$, set it equal to $ax^3 + 6x^2 + 9x + d$, and solve the resulting system.

$$(bx + c)^3 = b^3x^3 + 3b^2cx^2 + 3bc^2x + c^3$$

Equating the coefficients of terms of same exponent, we get $b^3 = a$, $3b^2c = 6$, $3bc^2 = 9$ and $c^3 = d$. Then the product $ad = b^3c^3$. We do not, however, have to solve for b and c (though one could). Instead, notice that using the two middle equations above, we have $3b^2c \times 3bc^2 = 6 \times 9$ which simplifies to $9b^3c^3 = 54$. Dividing through by 9 gives $b^3c^3 = 6$ and so $ad = \boxed{6}$.

40. At a Presidents' Day sale, you buy a scarf regularly priced at \$20. You also buy a sweater without a price tag which is on a 50% off rack. After using a 10% off your total purchase coupon, you pay \$45. Assuming there is no tax, what was the original price of the sweater?

Solution: Recall that a 10% discount means that you pay 90% of the total cost. Thus the equation for your total cost is $(20 + .5x) * .90 = 45$. Solving for x, we get $\boxed{\$60}$.

41. The parabola with the equation $y = ax^2+bx+c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $y = dx^2+ex+f$. Which of the following equal $a+b+c+d+e+f$?

Solution: Note first that reflecting $y = ax^2 + bx + c$ about $y = k$ will result in the parabola $y = dx^2 + ex + f$ which has the same vertex. Now, you should think about a reflection about $y = k$ in the following way: Reflect first about $y = 0$ (the x-axis). This will put the vertex at $(h, -k)$. Then shift this graph up $2k$ so that it has vertex (h, k). The equation for this graph is $y = -(ax^2 + bx + c) + 2k$. But this is the same as the graph of $y = dx^2 + ex^2 + f$. Thus $d = -a$, $e = -b$, and $f = -c + 2k$. Thus $a+b+c+d+e+f = a+b+c+-a+-b+-c+2k = 2k$. Thus the sum is $\boxed{2k}$.

42. Which of the following functions has a graph with exactly one hole, occurring when $x = 1$ and exactly one vertical asymptote, occurring when $x = -7$?

Solution: Recall that a hole occurs at $x = a$ if a is a zero of the denominator and the numerator. An asymptote occurs at $x = b$ if b is a zero of the denominator but not the numerator. Systematically factoring each of the answer choices shows that: (a) has a hole at $x = 1$ and an asymptote at $x = -7$, but also has a hole at $x = 0$; (b) has a hole at $x = -7$ and an asymptote at $x = 1$; (c) is the correct answer, with a hole at $x = 1$ and an asymptote at $x = -7$; (d) has holes at $x = -7$ and $x = 0$ with an asymptote at $x = 1$. Thus the correct answer is (c) $\boxed{f(x) = \frac{x^4-x^3}{x^2+6x-7}}$.

43. Let $S = 1 + 2 + 3 + \cdots + 10^n$, for n a positive integer. How many factors of 2 are there in the prime factorization of S?

Solution: We can use the summation formula to make this sum more compact: $S = \frac{10^n(10^n+1)}{2}$. The number 10^n is even, as it is divisible by 2, and $10^n + 1$ is odd. To see the number of factors of 2 in 10^n, we first will factor it:

$$10^n = (2 \cdot 5)^n = 2^n 5^n$$

Thus 10^n has n factors of 2. After dividing by 2, we have that S has $\boxed{n-1}$ factors of 2.

44. Amelia is driving at 60 mph, tapping her toe to a song at a rate of 5 taps per 3 seconds. She notices that every 10th tap always falls on a telephone pole. Assuming the telephone poles are always an equal distance apart on this road, how many miles apart are they?

Solution: Every 10th tap is every 6 seconds, since we have 5 taps every 3 seconds. Going 60mph means 60 miles takes 1 hour, which equates to one mile in one minute. Since 6 seconds is $\frac{1}{10}$th of a minute, the poles are every $\boxed{\frac{1}{10}}$ miles.

45. Find the value of k so that the function $f(x)$ is continuous.

$$f(x) = \begin{cases} kx^2 + 3 & x < 2 \\ k & x \geq 2 \end{cases}$$

Solution: Informally, for a function to be continuous, it needs to be able to be graphed without any holes, jumps, vertical asymptotes or other discontinuities of the graph. In order to avoid a jump at $x = 2$, we need both pieces evaluated at $x = 2$ to be the same value. That is, we need $kx^2+3 = k$, when $x = 2$. This gives the equation $4k + 3 = k$. Solving for k, we get $k = -1$, which is $\boxed{\text{None of These}}$.

More precisely, we require the limit from the left of $x = 2$ to be equal to the limit from the right of $x = 2$ and both to be equal to $f(2) = k$. That is,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = k$$

46. Let (a, b) be a point on the graph of $16x^2 + 36y^2 - 8x + 48y = 127$. What is the largest such b?

Solution: In an ellipse, the largest y value will be directly above the center. So first, we find the center by completing the square.

$$\begin{aligned} 16x^2 - 8x + +36y^2 + 48y &= 127 \\ 16(x^2 - \tfrac{1}{2}x) + 36(y^2 + \tfrac{4}{3}y) &= 127 \\ 16(x^2 - \tfrac{1}{2}x + \tfrac{1}{16}) + 36(y^2 + \tfrac{4}{3} + \tfrac{4}{9}) &= 127 + 1 + 16 \\ 16(x - \tfrac{1}{4})^2 + 36(y + \tfrac{2}{3})^2 &= 144 \\ \frac{(x-\frac{1}{4})^2}{9} + \frac{(y+\frac{2}{3})^2}{4} &= 1 \end{aligned}$$

Thus the center is $(\frac{1}{4}, -\frac{2}{3})$. The top-most y value on the graph will occur directly above the center, along the minor axis, which has length $2b = 2\sqrt{4} = 4$. So the distance from the center along the minor axis to the curve is 2. Thus $b = -\frac{2}{3} + 2 = \boxed{\frac{4}{3}}$.

47. Divide $1 - 7i$ by $6 - 2i$. Put your answer in $a + bi$ form.

Solution: To get the answer into $a + bi$ form, we must multiple both the numerator and denominator by the complex conjugate, $6 + 2i$, of the denominator.

$$\frac{1-7i}{6-2i}\left(\frac{6+2i}{6+2i}\right) = \frac{6-40i-14i^2}{36-4i^2} = \frac{6-40i+14}{36+4} = \frac{20-40i}{40} = \frac{1}{2} - i$$

Thus the answer is $\boxed{\frac{1}{2} - i}$.

48. Find the sum of all solutions to the equation

$$(x^3 + x^2 + 5x - 11)^2 - (4x^2 - 4x + 16)^2 = 0.$$

Solution: Notice that this equation is the difference of two squares. Recall that for any a, b, $a^2 - b^2 = (a + b)(a - b)$. We can factor and simplify the equation:

$$\begin{aligned}\Big((x^3 + x^2 + 5x - 11) + (4x^2 - 4x + 16)\Big)\Big((x^3 + x^2 + 5x - 11) - (4x^2 - 4x + 16)\Big) &= 0\\ \Big(x^3 + 5x^2 + x + 5\Big)\Big(x^3 - 3x^2 + 9x - 27\Big) &= 0\\ \Big(x^2(x+5) + 1(x+5)\Big)\Big(x^2(x-3) + 9(x-3)\Big) &= 0\\ (x^2+1)(x+5)(x^2+9)(x-3) &= 0\end{aligned}$$

The solutions to the four factors are, respectively, $i, -i, -5, 3i, -3i, 3$. Finally, the sum of all solutions is $\boxed{-2}$.

49. Two committees consisting of 3 and 5 people, respectively, are to be formed from a group of 8 people. Assuming no person is on both committees, in how many ways can this be done?

Solution: Since we are not putting anyone on multiple committees, after we choose 3 people from the 8, all of the remaining 5 will be forced to be on the second committee, so we need only account for the choice of the 3 person committee. The formula for this is

$$\binom{8}{3} = \frac{8!}{3!5!}$$

Therefore the answer is $\boxed{\frac{8!}{3!5!}}$.

50. Which of the following expressions is equivalent to

$$\log_2(x^3) - \log_2(xy^2) + 3\log_2(y^{1/2})?$$

Solution: First, using the Power Rule, move the constant multiple on the third term to the power to get $3\log_2(y^{1/2}) = \log_2(y^{3/2})$. Then, we can combine from left to right using the Quotient and Product rules:

$$\log_2(x^3) - \log_2(xy^2) + \log_2(y^{3/2}) = \log_2\left(\frac{x^3y^{3/2}}{xy^2}\right) = \log_2\left(\frac{x^2}{y^{1/2}}\right)$$

Thus the final answer is $\boxed{\log_2\left(\frac{x^2}{y^{1/2}}\right)}$.