The 37^{th} Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 24, 2018 at Regional Testing Centers
Second Round: April 14, 2018 at The University of Alabama at Birmingham

ALGEBRA II WITH TRIGONOMETRY EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

1. The mean of 25 numbers is 5. The mean of another 5 numbers is 35. What is the mean of all the 30 numbers?

(**A**) 7.5 (**B**) $\boxed{10}$ (**C**) 15 (**D**) 20 (**E**) None of these

2. There is exactly one integer a for which the polynomial $f(x) = ax^4 + 15x^3 - 5x^2 + 10x - a$ is divisible by $x + 3$. Find the sum of the value a and the coefficient on the x term of the quotient.

(**A**) $\boxed{10}$ (**B**) 16 (**C**) 20 (**D**) 24 (**E**) None of these

3. What is the shortest distance between the lines $y = 3x + 5$ and $2y - 6x = 8$?

(**A**) $\frac{2}{5}$ (**B**) 1 (**C**) $\frac{\sqrt{10}}{8}$ (**D**) $\boxed{\frac{\sqrt{10}}{10}}$ (**E**) None of these

4. Consider the sequence

$$1, -2, 3, -4, 5, -6, ..., n(-1)^{n+1}, ...$$

What is the sum of the first 1001 terms?

(**A**) 0 (**B**) -499 (**C**) 1001 (**D**) -1001 (**E**) $\boxed{\text{None of these}}$

5. A number is *abundant* if the sum of its proper divisors is greater than the number itself. Recall that a proper divisor of a number n is any positive divisor which is less than n. Which of the following is an abundant number?

(**A**) 16 (**B**) $\boxed{20}$ (**C**) 22 (**D**) 28 (**E**) 32

6. Find the number of solutions (a, b), with a, b real numbers, to the system of equations

$$\begin{cases} y + |x| = 3 \\ |x|y + x^3 = 0 \end{cases}$$

(**A**) 1 (**B**) $\boxed{2}$ (**C**) 3 (**D**) 5 (**E**) None of these

7. A contest has ten entries. How many ways are there to choose first, second, and third place, along with two unordered honorable mentions?

(**A**) 252 (**B**) 2520 (**C**) $\boxed{15{,}120}$ (**D**) 30,240 (**E**) None of these

8. Find the absolute value of the sum of all solutions to the equation $(1 - 2x)(x + 6) = 18$.

(**A**) 3.5 (**B**) $\boxed{5.5}$ (**C**) 12 (**D**) 13.5 (**E**) None of these

9. Which of the following functions is one-to-one on its domain?

(**A**) $f(x) = x^3 - x$ (**B**) $f(x) = x^2 + 2$ (**C**) $f(x) = e^{x^2}$ (**D**) $\boxed{f(x) = \sqrt{x + 4}}$ (**E**) $f(x) = x - \frac{1}{x}$

10. How many integers are excluded from the solution set of the inequality $\frac{3x - 2}{x} > 1$?

(**A**) Zero (**B**) One (**C**) $\boxed{\text{Two}}$ (**D**) Infinitely Many (**E**) None of these

11. For how many real values of x is the equation $(x + 2)^3 = x^3 + 8$ true?

(**A**) Zero (**B**) One (**C**) $\boxed{\text{Two}}$ (**D**) Infinitely Many (**E**) None of these

12. The rational expression $\dfrac{5x^2 - 2x + 3}{x^3 + 2x^2 + 5x + 10}$ is equivalent to the sum $\dfrac{A}{x+2} + \dfrac{Bx + C}{x^2 + 5}$. Find the product ABC.

(**A**) $\boxed{-36}$ (**B**) -495 (**C**) 0 (**D**) -30 (**E**) None of these

13. Find the range of the inverse function $f^{-1}(x)$ if $f(x) = \dfrac{x-3}{2x-8}$.

(**A**) $\left(\infty, \dfrac{3}{8}\right) \cup \left(\dfrac{3}{8}, \infty\right)$ (**B**) $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

(**C**) $(-\infty, 3) \cup (3, \infty)$ (**D**) $\boxed{(-\infty, 4) \cup (4, \infty)}$ (**E**) None of these

14. Find the maximum value of the function $f(x) = \dfrac{10}{4x^2 + 12x + 13}$.

(**A**) 2 (**B**) $\dfrac{1}{4}$ (**C**) $\dfrac{10}{13}$ (**D**) $\boxed{\dfrac{5}{2}}$ (**E**) None of these

15. The value of $\log 3.76$ to four decimal places is 0.5752. Find the value of $\log 37.6$.

(**A**) $\boxed{1.5752}$ (**B**) 3.627 (**C**) 5.752 (**D**) 10.5752 (**E**) None of these

16. Suppose for all positive integers n, we have $f(4 + n^2) = an + 2$ and $f(9 - n^2) = 3n - b$ for some numbers a and b. Then the value of $f(13)$ is

(**A**) $\boxed{-7}$ (**B**) -3 (**C**) 7 (**D**) not uniquely determined (**E**) not defined

17. Which of the following are true for all non-zero real values of x?

I. $\sin(3x) = 3\sin(x)$ II. $\dfrac{1}{x} + \dfrac{1}{2x} = \dfrac{1}{3x}$ III. $x^5(x^9) = x^{14}$

(**A**) I and II (**B**) I and III (**C**) I, II, and III (**D**) II only (**E**) $\boxed{\text{III only}}$

18. Find the number of distinct ordered pairs (x, y) satisfying $x^4y^4 - 13x^2y^2 + 40 = 4$, where x and y have integer values.

(**A**) 2 (**B**) 4 (**C**) 8 (**D**) $\boxed{16}$ (**E**) None of these

19. If a boy were 3 months more than 3/5 his current age, he would be 6.5 years old. How old is he currently?

(**A**) 5 years, 10 months (**B**) 15 years, 10 months

(**C**) $\boxed{\text{10 years, 5 months}}$ (**D**) 11 years, 3 months (**E**) None of these

20. Rationalize the denominator of $\dfrac{\sqrt[6]{125} + 7}{\sqrt[6]{125} - 1}$.

(**A**) $\boxed{2\sqrt{5} + 3}$ (**B**) -6 (**C**) 3 (**D**) $\dfrac{3\sqrt{5} - 1}{2}$ (**E**) None of these

21. The equation $x = ay^2 + by + c$ represents the graph of a horizontal parabola with x-intercept $(8, 0)$ and y-intercepts $(0, 2)$ and $(0, 4)$. Find $a + b + c$.

(**A**) -1 (**B**) $-\dfrac{2}{5}$ (**C**) $\dfrac{3}{5}$ (**D**) $\boxed{3}$ (**E**) None of these

22. Simplify the expression $x - (2y + x) - 2(3x - 5y + 3y)$.

(**A**) $\boxed{-6x + 2y}$ (**B**) $-6x + 14y$ (**C**) $-4x - 6y$ (**D**) $-4x + 14y$ (**E**) None of these

23. Find the sum of the absolute values of all solutions to $(x - 5)(x - 7)(x + 6)(x + 4) = 504$.

(**A**) 2 (**B**) $\boxed{20}$ (**C**) 201 (**D**) 2018 (**E**) None of these

24. Find the value of $f(x) = x^2 - x + 22$ when $x = 1 + 2i$.

(**A**) $\boxed{18 + 2i}$ (**B**) $18 - 2i$ (**C**) $18 + 6i$ (**D**) $22 - 6i$ (**E**) None of these

25. Find the sum of the squares of all values of x for which the matrix

$$\begin{bmatrix} 6x + 5 & 11 - 5x \\ -1 & 2x - 3 \end{bmatrix}$$

has a determinant of zero.

(**A**) $\frac{169}{9}$ (**B**) $\frac{211}{48}$ (**C**) $\frac{121}{144}$ (**D**) $\boxed{\frac{265}{144}}$ (**E**) None of these

26. How many distinct arrangements are there of the letters of the word CALCULUS?

(**A**) 120 (**B**) $\boxed{5040}$ (**C**) 6720 (**D**) 40320 (**E**) None of these

27. Freddie Freeman is teaching a clinic at a local tee-ball practice. Freddie crushes a ball demonstrating his home run swing. The ball's distance above the ground in feet, after t seconds, can be found using the equation $h = -16t^2 + 50t + 2.5$, and its distance in feet along the ground from home plate can be found using the equation $d = 96t$. Which of the following is closest to the distance the ball landed from home plate?

(**A**) 42 ft (**B**) 150 ft (**C**) $\boxed{305 \text{ ft}}$ (**D**) 347 ft (**E**) 411 ft

28. Find the product of all solutions to the equation $x^2 + x = 2 - 2\sqrt{x^2 + x - 2}$.

(**A**) -8 (**B**) $\boxed{-2}$ (**C**) 0 (**D**) 12 (**E**) None of these

29. The equation

$$-\frac{3}{5} + \frac{1 + 2x}{4} - \frac{8x - 7}{6} = \frac{1}{2}$$

has a single solution $x = a$. If $f(x) = -19x + 35$, find $f(\frac{1}{a})$ to the nearest integer.

(**A**) -31 (**B**) $\boxed{-15}$ (**C**) 28 (**D**) 43 (**E**) None of these

30. The function $x^4 + x^2 + 1$ factors into the product of two trinomials $(x^2 + ax + b)(x^2 + cx + d)$, where a, b, c, and d are integers. Find the sum $a + b + c + d$.

(**A**) $\boxed{2}$ (**B**) 4 (**C**) -4 (**D**) -2 (**E**) None of these

31. Suppose $f(x)$ is a degree six polynomial with positive leading coefficient, such that $f(-2) = -1$. What is the minimum number of real roots of the polynomial $f(x)$?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 4 (**E**) None of these

32. The binary operation $*$ between two integers m and n is defined by $m * n = m^2 + n^2$. Consider the following three statements:

I. The Commutative Law holds   II. The Associative Law holds   III. $m * m$ is even

Which of these statements is/are always true?

(A) I only **(B)** II only **(C)** III only **(D)** I and II **(E)** $\boxed{\text{I and III}}$

33. Find the largest integer value of n, with $0 \leq n \leq 100$, so that $(1 + i\sqrt{3})^n$ is a real number.

(A) 100 **(B)** $\boxed{99}$ **(C)** 98 **(D)** 97 **(E)** None of these

34. Which of the following is equal to $\dfrac{12^{40}}{36^{30}}$?

(A) $\boxed{\left(\dfrac{2}{3}\right)^{20}}$ **(B)** $\left(\dfrac{1}{3}\right)^{10}$ **(C)** $\left(\dfrac{1}{4}\right)^{20}$ **(D)** $3^9 4^{10}$ **(E)** 4^{10}

35. How many ordered 4-tuples of non-negative integers (a, b, c, d) satisfy $a + b + c + d = 17$?

(A) 68 **(B)** 560 **(C)** $\boxed{1140}$ **(D)** 5985 **(E)** None of these

36. If $g(x)$ is an even function, $h(x)$ is an odd function, and $f(x) = h(g(x)) - 2(h(x))^2$, what is true about the function $f(x)$?

(A) $\boxed{\text{Even}}$ **(B)** Odd **(C)** Neither even nor odd **(D)** Both even and odd **(E)** None of these

37. Let $g(x) = x^2 + 2x + 1$ and $f(x) = x^2 + 8x$. What is the value of the largest solution minus the smallest solution to the equation $(g \circ f)(x) = 9$?

(A) 2 **(B)** 11 **(C)** $4\sqrt{3}$ **(D)** $\boxed{6\sqrt{2}}$ **(E)** None of these

38. The equation $y^{-2} + 2y^{-1} - 15 = 0$ has two solutions a and b with $a > b$. Find $12a + 10b$.

(A) -14 **(B)** $-\dfrac{14}{15}$ **(C)** $\boxed{2}$ **(D)** 30 **(E)** None of these

39. A certain tax system charges 12% on the first \$10,000 of your salary, 15% on the next \$40,000 of your salary, and 20% on any portion of your salary which exceeds \$50,000. How much are the taxes on a salary of \$70,000?

(A) \$9,000 **(B)** $\boxed{\$11{,}200}$ **(C)** \$21,200 **(D)** \$32,900 **(E)** None of these

40. Given $f(x) = \dfrac{1}{x\sqrt{4x^2 + 9}}$, find an equivalent form for $f(x)$ in terms of trigonometric functions if $2x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(A) $\dfrac{2}{9}\cos\theta$ **(B)** $\dfrac{2}{9}\sin\theta$ **(C)** $\dfrac{1}{3}\cos\theta$ **(D)** $\boxed{\dfrac{2\cos^2\theta}{9\sin\theta}}$ **(E)** $\dfrac{2\sin^2\theta}{9\cos\theta}$

41. Find the number of solutions (a, b) to the equation below where both a and b are integers.

$$(x^2 - 2xy + y)^2 + 5x = y^2$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** $\boxed{\text{Infinitely Many}}$

42. Describe the roots of the following polynomial $f(x) = 6x^6 + 23x^5 + 3x^4 - 51x^3 - 189x^2 - 572x - 420$.

(**A**) 3 Real, 3 Complex (**B**) $\boxed{\text{4 Real, 2 Complex}}$ (**C**) 1 Real, 5 Complex (**D**) 2 Real, 2 Complex (**E**) 4 Real, 0 Complex

43. The system of equations below has two solutions (a, b) and (c, d) where $a, b, c,$ and d are real numbers. Find the distance between (a, b) and (c, d).

$$\begin{cases} x^2 - 2y^2 = 2 \\ xy = 2 \end{cases}$$

(**A**) $2\sqrt{3}$ (**B**) 4 (**C**) $\boxed{2\sqrt{5}}$ (**D**) 6 (**E**) None of these

44. The difference quotient is defined as $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$. Evaluate and simplify the difference quotient for $f(x) = \sqrt{x}$.

(**A**) 1 (**B**) $\dfrac{1}{2\sqrt{x}}$ (**C**) $\dfrac{\sqrt{h}}{h}$ (**D**) $\dfrac{2x}{\sqrt{x+h} - \sqrt{x}}$ (**E**) $\boxed{\dfrac{1}{\sqrt{x+h} + \sqrt{x}}}$

45. A particular math tournament wants to put the winning team names on the first, second, and third place trophies. However, since they won't know who wins until the very end, they plan to just buy enough trophies to cover any possible outcome. If there are 8 teams competing, what is the minimum number of trophies they will need to buy to ensure that the first, second, and third place teams each receive the appropriate place trophy with their name on it?

(**A**) 512 (**B**) 336 (**C**) 56 (**D**) $\boxed{24}$ (**E**) None of these

46. Find the 12th term in the arithmetic sequence whose first three terms are

$$\frac{\sqrt{3}+1}{5}, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}-1}{5}$$

(**A**) $\boxed{\dfrac{12\sqrt{3}-10}{5}}$ (**B**) $\dfrac{12\sqrt{3}-1}{5}$ (**C**) $\dfrac{13\sqrt{3}-11}{5}$ (**D**) $\dfrac{13\sqrt{3}+13}{5}$ (**E**) None of these

47. Megan and Sara begin jogging from the same point in opposite directions along the inside line of a circular track with a radius of 600 feet. Megan runs at 5 feet/second while Sara runs at 7 feet/second. How far has Megan run, in feet, when the two girls meet each other the first time? Round your answer to the nearest foot.

(**A**) 785 feet (**B**) $\boxed{\text{1571 feet}}$ (**C**) 3141 feet (**D**) 4712 feet (**E**) None of these

48. Find the value of the continued fraction

$$\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \ddots}}}}}$$

(**A**) $\boxed{\dfrac{\sqrt{15}-3}{2}}$ (**B**) $\dfrac{3}{2}$ (**C**) $\dfrac{1+\sqrt{29}}{14}$ (**D**) $\dfrac{-1+\sqrt{29}}{2}$ (**E**) None of these

49. Find the coefficient of the x^5 term in the expansion of $(2 + 3x)^8$.

(**A**) $2^3 3^5$ (**B**) $2^5 3^3$ (**C**) $\boxed{2^6 3^5 7}$ (**D**) $2^8 3^3 7$ (**E**) None of these

50. For how many integers n between 1 and 200 does $x^2 + 2x - n$ factor into the product of two linear factors with integer coefficients and constant terms?

(**A**) 12 (**B**) $\boxed{13}$ (**C**) 14 (**D**) 15 (**E**) None of these