The 33rd Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 22, 2014 at Regional Testing Centers
Second Round: March 15, 2014 at The University of North Alabama

COMPREHENSIVE EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\widehat{AB}$ is the arc between A and B $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptoanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

The following information is courtesy of the U.S. Bureau of Labor Statistics.

- The median salary of a Mathematician in 2012 was $101,360 per year.
- Over the next 10 years, the job opportunities for mathematicians are expected to grow by 23%!

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

1. Segment $\overline{AB}$ is a diameter of circle O. P is a point on the circumference such that $AP = 7$ and $BP = 6$. What is the area of circle O?

(**A**) $\frac{21\pi}{2}$ (**B**) $\boxed{\frac{85\pi}{4}}$ (**C**) 42π (**D**) 85π (**E**) $\frac{42\pi}{3}$

2. Consider the ellipse given by the equation $9x^2 + 4y^2 - 18x + 16y - 11 = 0$. Let c be the length of the major axis and d be the length of the minor axis. Find cd.

(**A**) 36 (**B**) $\boxed{24}$ (**C**) 6 (**D**) 16 (**E**) 18

3. How many distinguishable permutations are there of the letters in the word CIRCLE?

(**A**) 60 (**B**) 720 (**C**) $\boxed{360}$ (**D**) 120 (**E**) 240

4. What is the area of a circle inscribed in a dodecagon with an apothum of length 13?

(**A**) 72π (**B**) $\boxed{169\pi}$ (**C**) 81π (**D**) 121π (**E**) 144π

5. The following system has a unique solution (a, b) satisfying it.

$$\begin{cases} \frac{1}{x} + \frac{3}{y} = 7 \\ \frac{4}{x} - \frac{2}{y} = 1 \end{cases}$$

Find $\frac{1}{a} + \frac{1}{b}$.

(**A**) $\boxed{\frac{22}{7}}$ (**B**) $\frac{57}{7}$ (**C**) $\frac{62}{5}$ (**D**) $\frac{6}{5}$ (**E**) $\frac{78}{7}$

6. The length of a rectangle is diminished by 10 feet, while the width is increased by 6 feet. If the result is a square whose area is equal to that of the original rectangle, what is the area?

(**A**) 196 feet^2 (**B**) 256 feet^2 (**C**) 169 feet^2 (**D**) $\boxed{225 \text{ feet}^2}$ (**E**) 349 feet^2

7. Which of the following angles satisfies $\cot\theta\cos\theta < 0$?

(**A**) $450°$ (**B**) $\boxed{\frac{27\pi}{5}}$ (**C**) $-\frac{\pi}{2}$ (**D**) $\frac{4\pi}{7}$ (**E**) $-189°$

8. A total of 28 handshakes were exchanged at the conclusion of a party. Assuming that each participant was equally polite towards all of the others, the number of people present was

(**A**) 14 (**B**) 28 (**C**) 56 (**D**) $\boxed{8}$ (**E**) 7

9. If the radius of a circle is increased by 100%, the area is increased by:

(**A**) 100% (**B**) 200% (**C**) $\boxed{300\%}$ (**D**) 400% (**E**) 800%

10. Which of the following graphs intersect the graph of $y = x^2 - 4$ the most times?

(**A**) $x^2 + y^2 = 1$ (**B**) $y = x + 5$ (**C**) $y = -x^2 + 4$ (**D**) $y = x^3$ (**E**) $\boxed{x^2 + y^2 = 9}$

11. The sides of a triangle are 14, 16 and 18. Determine the length of the shortest altitude.

(A) $\frac{24\sqrt{5}}{9}$ **(B)** $4\sqrt{5}$ **(C)** $\boxed{\frac{48\sqrt{5}}{9}}$ **(D)** $8\sqrt{5}$ **(E)** $\frac{64\sqrt{5}}{9}$

12. Define a function as follows: for $x > 0$, $f(x) = f(x-2)$ and for $x \le 0$, $f(x) = |x|$. Find the value of $f(2.7) - f(5)$.

(A) 2.3 **(B)** $-.7$ **(C)** -2.3 **(D)** $\boxed{.3}$ **(E)** 1.7

13. The perimeter of a right triangle is 42 and the sum of the squares of its sides is 722. Determine the area of the right triangle.

(A) 100 **(B)** 64 **(C)** 81 **(D)** $\boxed{42}$ **(E)** 121

14. A fair 10-sided die is rolled. What is the probability a prime number was rolled?

(A) $\frac{1}{2}$ **(B)** $\boxed{\frac{2}{5}}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{7}{10}$

15. The product of the roots of $x^2 - 4\sqrt{7} + \dfrac{28}{x^2}$ is

(A) -14 **(B)** $4\sqrt{7}$ **(C)** $-4\sqrt{7}$ **(D)** $2\sqrt{7}$ **(E)** $\boxed{-2\sqrt{7}}$

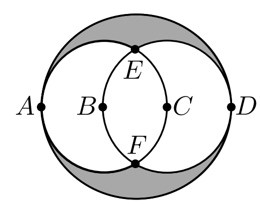
16. How many solutions are there to the equation $\left|2x^2 - x - 1\right| = x$?

(A) 0 **(B)** 1 **(C)** $\boxed{2}$ **(D)** 3 **(E)** 4

17. If $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$ for $k \neq 0$, then $r =$

(A) $-\log_2 5$ **(B)** $\log_5 2$ **(C)** $\log_{10} 5$ **(D)** $\boxed{\log_2 5}$ **(E)** $5/2$

18. In the figure shown, two small circles are inscribed and tangent to the larger circle at points A and D. Let points B and C be the centers of the circles, where $AB = BC = CD = 1$. Find the area of the shaded region.



(A) $\dfrac{5\pi}{12} - \dfrac{\sqrt{3}}{2}$ **(B)** $\dfrac{13\pi}{24} - \dfrac{\sqrt{3}}{2}$ **(C)** $\dfrac{7\pi}{12} - \dfrac{\sqrt{3}}{2}$

(D) $\dfrac{11\pi}{24} - \dfrac{\sqrt{3}}{2}$ **(E)** $\boxed{\dfrac{11\pi}{12} - \dfrac{\sqrt{3}}{2}}$

19. One of the zeros of the function $f(x) = x^4 + 5x^3 + 10x^2 + 20x + 24$ is $2i$, where $i = \sqrt{-1}$. What is the sum of the the real zeros of $f(x)$?

(A) $\boxed{-5}$ **(B)** 4 **(C)** 0 **(D)** 5 **(E)** -4

20. How many integers satisfy $n^4 + 6n < 6n^3 + n^2$?

(A) 8 **(B)** $\boxed{4}$ **(C)** 0 **(D)** 5 **(E)** Infinitely many

21. The sides of a regular polygon of n sides, $n > 4$ are extended to form a star. The number of degrees at each point of the star is:

(**A**) $\frac{360}{n}$ (**B**) $\boxed{\frac{(n-4)180}{n}}$ (**C**) $\frac{(n-2)180}{n}$ (**D**) $180 - \frac{90}{n}$ (**E**) $\frac{180}{n}$

22. If $\sin t = a$, $\cos t = b$ and $\tan t = c$, then the expression

$$\sin(-t + 4\pi) + 3\cos(\frac{\pi}{2} - t) - \tan(t - 3\pi)$$

can be written as

(**A**) $a - 3b - c$ (**B**) $-a + 3b + c$ (**C**) $-a + 3b + \frac{c}{2}$ (**D**) $\boxed{2a - c}$ (**E**) $4a + c$

23. Find the sum of all distinct three digit numbers that contain only the digits 1, 2, 3, 4, each at most once.

(**A**) 1250 (**B**) 1110 (**C**) $\boxed{6660}$ (**D**) 3230 (**E**) 8670

24. Find the area of a triangle if two adjacent sides of 15 and 8 include an angle of 150°.

(**A**) 60 (**B**) 120 (**C**) $30\sqrt{2}$ (**D**) $\boxed{30}$ (**E**) $60\sqrt{2}$

25. There exist postive integers A, B and C, with no common factor greater than 1, such that

$$A \log_{200} 5 + B \log_{200} 2 = C$$

What is $A + B + C$?

(**A**) $\boxed{6}$ (**B**) 7 (**C**) 8 (**D**) 9 (**E**) 10

26. Henry starts a trip when the hands of the clock are together between 8 AM and 9 AM. He arrives at his destination between 2 PM and 3 PM when the hands are exactly 180° apart. The trip takes

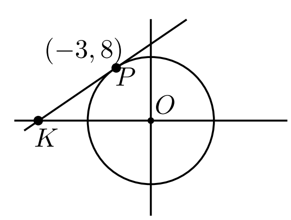
(**A**) 5 hr. 43 min. (**B**) $\boxed{\text{6 hr.}}$ (**C**) 6 hr. 43 min. (**D**) 5 hr. 17 min. (**E**) 6 hr. 30 min.

27. In the rhombus $ABCD$, diagonal DB is 20 units longer than diagonal AC. If the area of the rhombus is 150 u^2, what is the length of the longest diagonal?

(**A**) $\boxed{\text{30 u}}$ (**B**) 10 u (**C**) 50 u (**D**) 35 u (**E**) 15 u

28. In the figure shown, a line is tangent to the circle at the origin. The point of tangency is $(-3, 8)$. The line intersects the x-axis at $x = k$. Find k.

(**A**) $\boxed{-\frac{73}{3}}$ (**B**) $-\frac{8}{3}$ (**C**) $-\frac{64}{3}$ (**D**) $-\frac{16}{3}$ (**E**) $-\frac{32}{3}$



29. A group of seven people are waiting for a taxi. The taxi only holds four people. In how many ways can they pick which four people get in the taxi?

(**A**) 210 (**B**) $\boxed{35}$ (**C**) 840 (**D**) 24 (**E**) 720

30. Find the product of the real solutions to the equation $(x^2 - x + 1)(x^2 - x + 2) = 12$.

(**A**) $\sqrt{21}$ (**B**) 0 (**C**) $\boxed{-2}$ (**D**) -3 (**E**) 6

31. Find the product of all solutions of the equation $\sin^2 x \sec x + 2\sin^2 x = \sec x + 2$ in the interval $[0, 2\pi)$.

Problem thrown out. The correct answer is $\dfrac{8\pi^2}{9}$.

(**A**) $\dfrac{2\pi^4}{3}$ (**B**) $\dfrac{35\pi^4}{48}$ (**C**) 0 (**D**) $\dfrac{4\pi^3}{9}$ (**E**) $\dfrac{35\pi^3}{72}$

32. Find the seventh term of the geometric progression 36, -12, 4, ...

(**A**) -36 (**B**) $\boxed{\frac{4}{81}}$ (**C**) $\frac{23}{16}$ (**D**) 24 (**E**) $-\frac{4}{27}$

33. The angle of a sector of a circle is 96° and its corresponding arc is 24 inches. Find the area of the circle.

(**A**) $\dfrac{1950}{\pi}$ (**B**) $\dfrac{1800}{\pi}$ (**C**) $\boxed{\dfrac{2025}{\pi}}$ (**D**) $\dfrac{2175}{\pi}$ (**E**) $\dfrac{1845}{\pi}$

34. Evaluate $\sin 80^\circ \cos 65^\circ - \cos 80^\circ \sin 65^\circ$.

(**A**) $\dfrac{\sqrt{1-\sqrt{3}}}{2}$ (**B**) $\dfrac{\sqrt{2+\sqrt{3}}}{2}$ (**C**) $\boxed{\dfrac{\sqrt{2-\sqrt{3}}}{2}}$ (**D**) $\dfrac{\sqrt{1+\sqrt{3}}}{2}$ (**E**) $\dfrac{\sqrt{2-2\sqrt{3}}}{2}$

35. Simplify the expression $\left(\sqrt{2} + \sqrt{2}i\right)^{2014}$.

(**A**) -4^{2014} (**B**) -4^{1007} (**C**) $\boxed{-2^{2014}i}$ (**D**) 2^{2014} (**E**) $2^{1007}i$

36. In a triangle, one angle is three times as large as the other and the third is 20° greater than the sum of the other two. What are the measures of the angles of the triangle?

(**A**) $5^\circ, 15^\circ, 160^\circ$ (**B**) $10^\circ, 30^\circ, 140^\circ$ (**C**) $\boxed{20^\circ, 60^\circ, 100^\circ}$ (**D**) $25^\circ, 75^\circ, 80^\circ$ (**E**) $30^\circ, 60^\circ, 90^\circ$

37. Before Ashley started a 2-hour drive, her car's odometer reading was 27972, a palindrome (A *palindrome* is a number that reads the same backwards as forwards.) At her destination, the odometer reading was another palindrome. If Ashley never exceeded 75 mph, which of the following was her average speed?

(**A**) 50 mph (**B**) $\boxed{\text{55 mph}}$ (**C**) 60 mph (**D**) 65 mph (**E**) 70 mph

38. How many different prime numbers are factors of N if

$$\log_2(\log_3(\log_5(\log_7 N))) = 11$$

(**A**) $\boxed{1}$ (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 7

39. The equation $4x^2 - y^2 = 11$ has exactly one integer pair solution (x, y) with both $x, y > 0$. Find x.

(**A**) 5 (**B**) -5 (**C**) -3 (**D**) $\boxed{3}$ (**E**) 0

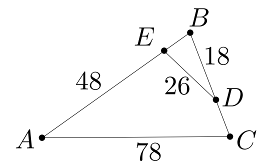
40. Find the area of a rhombus if one diagonal is 30 and a side is 17.

(**A**) [240] (**B**) 510 (**C**) 255 (**D**) 450 (**E**) 170

41. A sequence is defined recursively as $a_1 = 2$ and $a_k = -2a_{k-1} + 3$ where $k \geq 2$. What is the median of the first six terms of the sequence?

(**A**) -1 (**B**) -2.5 (**C**) [0.5] (**D**) 5 (**E**) 2

42. In the given figure, $CD = BE$ and $\angle BDE = \angle BAC$. Find the perimeter of triangle ABC.
Problem thrown out. There is not a consistent scale factor for the triangles, so with the numbers provided they are not actually similar.



(**A**) 135 (**B**) 144 (**C**) 152 (**D**) 160 (**E**) 162

43. The ratio of the area of a square inscribed in a semicircle to the area of a square inscribed in the entire circle is

(**A**) 1:2 (**B**) 2:3 (**C**) [2:5] (**D**) 3:4 (**E**) 3:5

44. If $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find $\cos(3x)$.

(**A**) $\frac{14}{25}$ (**B**) $\frac{12}{5}$ (**C**) $-\frac{36}{125}$ (**D**) $\boxed{-\frac{44}{125}}$ (**E**) $\frac{4}{5}$

45. For exactly two real values of m, m_1 and m_2, the line $y = mx+3$ intersects the parabola $y = x^2+2x+7$ at exactly one point. Compute $m_1^2 + m_2^2$.

(**A**) 8 (**B**) [40] (**C**) 25 (**D**) 24 (**E**) 68

46. If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$ then $f(3)$

(**A**) -5 (**B**) -2 (**C**) 1 (**D**) 3 (**E**) [8]

47. If $xy = 2$, $yz = 3$ and $xz = 5$, what is the value of $x^2 + y^2 + z^2$?

(**A**) $\boxed{\frac{361}{30}}$ (**B**) 20 (**C**) $\frac{6}{5}$ (**D**) $\frac{65}{6}$ (**E**) 32

48. There are four regular triangular pyramids lined up in a row. The first three pyramids have side lengths of 6, 8 and 10. The volume of the last pyramid is the sum of the volumes of the first three. Determine the side length of the fourth pyramid.

(**A**) 18 (**B**) 4 (**C**) 16 (**D**) 8 (**E**) [12]

49. The equation $x + \sqrt{x-2} = 4$ has

(**A**) 2 real roots (**B**) 1 real and 1 imaginary root

(**C**) 2 imaginary roots (**D**) no roots (**E**) [1 real root]

50. Define a function to be "nice" if $f(x+y) = f(x) + f(y)$ for any values of x and y in the domain. Which of the following functions is nice on its domain?

(**A**) $f(x) = \ln(x)$ (**B**) $f(x) = \sin(x)$ (**C**) $f(x) = \sqrt{x}$ (**D**) $\boxed{f(x) = 5x}$ (**E**) $y = |x|$