The 45th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

ACTM

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ALGEBRA II EXAMINATION

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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions are not arranged in order of difficulty. For each question, choose the best of the five options labeled A, B, C, D and E. Calculators are NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a paper with: all questions answered correctly earns a score of 250, all questions left blank earns a score of 50, and all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Analysis
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, Northrop Grumman, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Determine the sum of all distinct solutions to the equation

$$(x+1)^4 - 2(x+1)^2 + 1 = 0.$$

(**A**) 0 (**B**) -2 (**C**) -4 (**D**) -6 (**E**) None of these

Solution: We complete the square in terms of $(x+1)^2$ to get

$$(x+1)^2 = 1.$$

This gives $x = 0$ or $x = -2$.

Answer: B) -2

2. Suppose x and y are consecutive integers such that $x^3 - y^3 = 721$ and $x^2 - y^2 = 31$. Find the product xy.

(**A**) 160 (**B**) 200 (**C**) 225 (**D**) 240 (**E**) None of these

Solution:

Using the factorizations of $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ and $x^2 - y^2 = (x-y)(x+y)$, note that $x - y = 1$ and $(x+y)^2 - (x^2+xy+y^2) = xy = 31^2 - 721 = 961 - 721 = 240$. **Answer: D)** 240**.**

3. What is the smallest x value of the intersection points of the graphs of $f(x) = 10 - |x-1|$ and $g(x) = |x+2|$?

(**A**) $-\frac{11}{2}$ (**B**) $-\frac{7}{2}$ (**C**) $\frac{7}{2}$ (**D**) $\frac{11}{2}$ (**E**) None of these

Solution:

The left-most parts of the functions ($x < 1$ for $f(x)$, $x < -2$ for $g(x)$) intersect at

$$\begin{aligned} 10 + x - 1 &= -x - 2 \\ 2x &= -11 \\ x &= -11/2. \end{aligned}$$

Answer: A) $-\frac{11}{2}$**.**

4. A complex-valued function f which is defined on the integers is called a **character** if $f(0) = 1$, $|f(n)| = 1$ and

$$f(n+m) = f(n)f(m)$$

for any integers n, m. Suppose f is a character and $f(1) = -\frac{1}{2} + bi$ for some positive number b. If $f(2026) = x + yi$ for real numbers x and y, determine x.

(**A**) $-\frac{1}{2}$ (**B**) $-\frac{\sqrt{3}}{2}$ (**C**) $\frac{\sqrt{3}}{2}$ (**D**) $\frac{1}{2}$ (**E**) None of these

Solution: Since $|f| = 1$, we have $b = \pm\frac{\sqrt{3}}{2}$. So, $f(1) = e^{\frac{2\pi i}{3}}$. (You can also start taking powers and notice that it's a cube root of 1.) Take 2026 by 3 and get a remainder of 1. Hence,

$$f(2026) = (f(1))^{2026} = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + yi$$

where $y = \sqrt{3}2$.

Answer: C) $\dfrac{\sqrt{3}}{2}$

5. Determine the number of real pairs (x, y) which satisfy the system of equations

$$\begin{cases} x^2 + \dfrac{2x}{y} + \dfrac{1}{y^2} - 1 = 0, \\ xy + 1 = 0. \end{cases}$$

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

Solution: We compute the square with the first equation to get

$$(x + \frac{1}{y})^2 = 1 \implies (xy + 1)^2 = y^2.$$

Combining with the second equation, we get $y = 0$. However, the first equation is not defined for $y = 0$.

Answer: E) NOTA

6. We say two real-valued functions u, v defined on the positive real numbers are Legendre conjugates if

$$u(x) + v(y) \geq xy$$

for any $x, y > 0$. Suppose u is Legendre conjugate to itself. Which of the following could be the function $u(x)$?

(**A**) x (**B**) $\dfrac{x^2}{2}$ (**C**) $\dfrac{3}{7}x^2$ (**D**) $\sqrt{x}$ (**E**) None of these

Solution: Testing the condition with $x = y$ gives $u(x) \geq \frac{1}{2}x^2$. Thus, the only possibility of the choices are B) or C). To show that B) is possible, use AM-GM or just rewrite it as a square $(x - y)^2 \geq 0$.

Answer: B) $\dfrac{x^2}{2}$

7. Let N be the smallest positive integer such that

$$\prod_{n=2}^{N} \left[\log_{2026}(n) \log_{n+1}(2026)\right] < \frac{1}{10}.$$

Find the sum of the digits of N.

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) None of these

Solution: Use the change of base formula to simplify this to

$$\frac{\log 2}{\log(N+1)} < \frac{1}{10} \iff N + 1 > 2^{10}.$$

So, $N = 1024$.

Answer: C) 7

8. Alice has \$2.00 in quarters and dimes. Bob has \$1.60 in nickels and dimes. Suppose Alice and Bob have the same number of dimes and the same total number of coins. How many dimes do they have together?

(**A**) 18 (**B**) 22 (**C**) 26 (**D**) 30 (**E**) None of these

Solution: Let Q be the number of quarters and D the number of dimes that Alice has. The condition tells us that Bob also has D dimes and Q nickels. This sets up the system of equations

$$\begin{cases} 25Q + 10D = 200 \\ 5Q + 10D = 160. \end{cases}$$

We solve this to get $Q = 2$, $D = 15$.

Answer:D) 30

9. The quadratic function $p(x) = mx^2 + 5x + 2$ has at most one real solution. What is the range of the possible values of m?

(**A**) $\left[\frac{25}{8}, \infty\right) \cup \{0\}$ (**B**) $\left[\frac{25}{8}, \infty\right)$ (**C**) $[0, \infty)$ (**D**) $\left(-\infty, \frac{25}{8}\right]$ (**E**) None of these

Solution: To have a real solution, either the discriminant of the quadratic is not positive or p is linear ie. $m = 0$.

Answer: A) $[\frac{25}{8}, \infty) \cup \{0\}$

10. Suppose $Q(x) = x^2 + bx + c$ has a root at $x = 2$ and $Q(4) = 10$. Find $Q(-4)$.

(**A**) 6 (**B**) 12 (**C**) 18 (**D**) 24 (**E**) None of these

Solution: We can factor $Q(x) = (x - 2)(x + a)$. Then,

$$10 = Q(4) = 2(a + 4) \implies a = 1.$$

Hence, $Q(-4) = (-6)(-3) = 18$.

Answer: C) 18

11. Simplify the expression

$$256^{-\frac{1}{8}} \left(\frac{\left(\frac{a^2}{25}\right)^2}{c^{12}} \right)^{\frac{1}{4}} \left(\frac{b}{10c^2} \right)^{-2}.$$

(**A**) $10ab^{-2}c$ (**B**) $\frac{25}{2}a^2bc^2$ (**C**) $20ab^{-1}c^{-1}$ (**D**) $25a^2bc^3$ (**E**) None of these

Solution: We simplify each term

$$\frac{1}{2} \cdot \frac{a}{5c^3} \cdot \frac{100c^4}{b^2} = \frac{10ac}{b^2}.$$

Answer: A)$10ab^{-2}c$

12. Let r, s, t be the three distinct roots of the equation $4x^3 - 12x^2 - 3x + 6 = 0$. Compute

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t}.$$

(**A**) $\frac{1}{2}$ (**B**) $\frac{3}{4}$ (**C**) $\frac{5}{6}$ (**D**) $\frac{11}{12}$ (**E**) None of these

Solution: By Vieta's formulas, we have $\frac{-3/4}{-6/4} = \frac{1}{2}$.

Answer: A) $\dfrac{1}{2}$

13. Two parallel lines have slope 4 and y-intercepts 3 and -2. Find the shortest distance between these two lines.

(**A**) 5 (**B**) $\dfrac{5}{\sqrt{7}}$ (**C**) $\dfrac{5}{\sqrt{17}}$ (**D**) $\dfrac{7}{\sqrt{5}}$ (**E**) None of these

Solution: One of the lines is $y = 4x+3$ and the other is $y = 4x-2$. The perpendicular line passing through $(0,3)$ is $y = -\frac{1}{4}x + 3$ and finding its intersection with $y = 4x - 2$ we find that

$$-\frac{1}{4}x + 3 = 4x - 2$$

$$5 = \frac{17}{4}x \implies x = \frac{20}{17}, y = \frac{46}{17}$$

Therefore, the distance is $\frac{1}{17}\sqrt{20^2+5^2} = \frac{5}{\sqrt{17}}$.

Answer: C) $\dfrac{5}{\sqrt{17}}$

14. The solution set of the inequality $\dfrac{2x-1}{x+5} < 1$ consists of one open interval. Find the length of this interval.

(**A**) 8 (**B**) 9 (**C**) 10 (**D**) 11 (**E**) None of these

Solution: We subtract 1 on both sides to get

$$\frac{2x-1-x-5}{x+5} = \frac{x-6}{x+5} < 0.$$

So, the numerator and denominator must be opposite signs. Hence, $-5 < x < 6$.

Answer: D) 11

15. What is the coefficient of the x^9y^{10} term in the expansion of $(x+y)^8(2x-y^2)^7$?

(**A**) -7840 (**B**) 7840 (**C**) -8630 (**D**) 8630 (**E**) None of these

Solution: We determine which terms contribute and then apply the binomial theorem. Let x^ay^{8-a} from the first binomial and $x^b(y^2)^{7-b}$ from the second binomial. This gives

$$\begin{cases} a+b=9 \\ (8-a)+(14-2b)=10. \end{cases}$$

Thus, we have $b=3$ and $a=6$. Thus, we compute

$$\binom{8}{6}\cdot\binom{7}{3}(2)^3(-1)^4 = 7840.$$

Answer: B) 7840

16. Solve the equation $2\log_{36}(x) + 3\log_{216}(x+1) = 1$.

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 6 (**E**) None of these

Solution: Using the change of base formula and adding, we get $\log_6(x(x+1)) = 1 \implies x^2 + x - 6 = 0$. Hence, $x = 2, -3$.

Answer: B) 2

17. Suppose the complex number $z = a + bi$ satisfies $(a+2bi)^2 = z(a-bi)$. Which statement is necessarily true?

(**A**) z is real (**B**) z is purely imaginary (**C**) z has modulus 1 (**D**) $z = 0$ (**E**) None of these

Solution: We get $(a^2 - 4b^2) + (4ab)i = a^2 + b^2$. Matching real and imaginary parts, $b = 0$ and the equation is satisfied for any a.

Answer: A) z **is real.**

18. Find a solution of the equation $\dfrac{3^{x-2}}{2} = 3^{-1} + 3^{-(x+2)}$.

(**A**) $\log_3(3 - \sqrt{11})$ (**B**) $\log_3(3 - \sqrt{7})$ (**C**) $\log_3(3 + \sqrt{7})$ (**D**) $\log_3(3 + \sqrt{11})$ (**E**) None of these

Solution:

$$\begin{aligned}
\frac{3^{x-2}}{2} &= \frac{1}{3} + 3^{-(x+2)} \\
3^{x-2} &= \frac{2}{3} + 2 \cdot 3^{-(x+2)} \\
3^{2x} &= \frac{2}{3} \cdot 3^{x+2} + 2 \\
3^{2x} &= 6 \cdot 3^x + 2 \\
3^{2x} - 6 \cdot 3^x - 2 &= 0 \\
3^x &= \frac{6 \pm \sqrt{36+8}}{2} \\
3^x &= 3 + \sqrt{11} \text{(discard negative solution)} \\
x &= \log_3(3 + \sqrt{11})
\end{aligned}$$

Answer: D) $\log_3(3 + \sqrt{11})$**.**

19. Suppose $\ell_1, \ell_2, \ell_3, \ell_4$ are non-vertical lines in the Cartesian plane (the xy-plane) such that the odd indexed lines are parallel. Assume that m_i is the slope of line ℓ_i for $i = 1, 2, 3, 4$, and that the slopes satisfy

$$m_1 m_2 m_3 m_4 = 1, \quad m_1 m_2 = m_3 m_4, \quad (-1)^k m_k > 0 \text{ for } k = 1, 2, 3, 4.$$

If ℓ_1 has equation $2x + 3y = 6$, determine the slope of ℓ_4.

(**A**) $\frac{2}{3}$ (**B**) $-\frac{2}{3}$ (**C**) $\frac{3}{2}$ (**D**) $-\frac{3}{2}$ (**E**) None of these

Solution:

The inequality asserts that m_1, m_3 are negative and m_2, m_4 are positive. By the first and second equations,

$$\begin{aligned} m_3 &= \frac{1}{m_1 m_2 m_4} \\ m_3 &= \frac{m_1 m_2}{m_4} \\ \frac{1}{m_1 m_2 m_4} &= \frac{m_1 m_2}{m_4} \\ m_1^2 &= \frac{1}{m_2^2} \\ m_1 &= -\frac{1}{m_2} \end{aligned}$$

Therefore, $m_3 m_4 = -1$, i.e., m_3 and m_4 are perpendicular. Since $m_1 = m_3$ (by the assumption that ℓ_1, ℓ_3 are parallel), $m_4 = -\frac{1}{m_1}$. Finally, since $m_1 = -\frac{2}{3}$, $m_4 = \frac{3}{2}$.

Answer: C) $\frac{3}{2}$**.**

20. Simplify the following:

$$(x^2 - 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1).$$

(**A**) x^{10} (**B**) $x^{10} - 1$ (**C**) $x^{10} + x^8 + x^6 + x^4 + x^2 + 1$
(**D**) $x^{10} - x^8 + x^6 - x^4 + x^2 - 1$ (**E**) None of these

Solution:

The inequality asserts that m_1, m_3 are negative and m_2, m_4 are positive. By the first and second equations,

$$\begin{aligned} &(x^2 - 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1) \\ &= (x - 1)(x^4 + x^3 + x^2 + x + 1) \cdot \\ &\quad (x + 1)(x^4 - x^3 + x^2 - x + 1) \\ &= (x^5 - 1)(x^5 + 1) \\ &= x^{10} - 1 \end{aligned}$$

Answer: B) $x^{10} - 1$**.**

21. Which of the following polynomials $f(x)$ satisfies $f(3 + x) = f(3 - x)$ for all x?

(**A**) $f(x) = x^2 - 9$ (**B**) $f(x) = x^2 + 6x + 9$ (**C**) $f(x) = (x^2 - 6x + 9)^3$
(**D**) $f(x) = (x^3 + 3x^2 + 6x + 9)^2$ (**E**) None of these

Solution:

This property is symmetry about the line $x = 3$. We can check specific values, like $x = 3$, for which (A) gives the false statement $-9 = 36 - 9$, (B) gives the false statement $9 = 36 + 36 + 9$, (C) gives the true statement $9^3 = 9^3$, and (D) gives the false statement $9^2 = (6^3 + 3 \cdot 36 + 36 + 9)^2$.

Answer: C) $f(x) = (x^2 - 6x + 9)^3$**.**

22. Which of the following polynomials has NO real roots?

(**A**) $x^2 + 8x + 16$ (**B**) $x^4 + 8x^2 + 16$ (**C**) $x^8 - 8x^4 + 16$ (**D**) $x^9 + x^2 + 99$ (**E**) None of these

Solution: (B) factors as $(x^2+4)^2$, which has no real roots. The other polynomials have either odd degree or have factors with real roots $[(x+4)(x+4)$,$(x^4-4)(x^4-4)]$

Answer: B) x^4+8x^2+16**.**

23. Let X be a closed interval such that the functions $f(x) = x^2 - 6x + 5$ and $g(x) = x^3 + 1$ are both invertible when restricted to X, but $h(x) = x^2 - 4x - 7$ is not invertible when restricted to X. Which of the following intervals are possible candidates for X?

(**A**) $[1, 3]$ (**B**) $[1, 4]$ (**C**) $(1, 4)$ (**D**) $[2, 4]$ (**E**) None of these

Solution:

The function $g(x)$ is invertible over all real numbers. The inverses of f and h depend on the location of their vertices $x = 3, 2$, respectively. X cannot contain the x-value of the vertex of f in its interior but must contain the x-value of the vertex of h in its interior.

Answer: A) $[1, 3]$**.**

24. Suppose n is a real number such that $n + \frac{1}{n} = 3$. Compute the value of $n^3 + \frac{1}{n^3}$.

(**A**) 9 (**B**) 18 (**C**) 24 (**D**) 27 (**E**) None of these

Solution:

$$\begin{aligned}
\left(n + \frac{1}{n}\right)^2 &= 9 \\
n^2 + 2 + \frac{1}{n^2} &= 9 \\
n^2 + \frac{1}{n^2} &= 7 \\
\left(n^2 + \frac{1}{n^2}\right)\left(n + \frac{1}{n}\right) &= 7 \cdot 3 \\
n^3 + \left(\frac{1}{n} + n\right) + \frac{1}{n^3} &= 21 \\
n^3 + \frac{1}{n^3} &= 18
\end{aligned}$$

Answer: B) 18**.**

25. Find the value of

$$\sum_{n=1}^{99} \frac{1}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}}.$$

(**A**) $3\sqrt{5}$ (**B**) 9 (**C**) $10\sqrt{2}$ (**D**) 26 (**E**) None of these

Solution:

$$
\begin{aligned}
&= \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots \frac{1}{\sqrt{99}+\sqrt{100}} \\
&= \frac{\sqrt{2}-1}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \cdots \frac{\sqrt{100}-\sqrt{99}}{100-99} \\
&10 - 1 = 9.
\end{aligned}
$$

Answer: B) 9.

26. Which of the following is the multiplicative inverse of $\begin{pmatrix} 7 & 2 \\ 11 & 3 \end{pmatrix}$?

(**A**) $\begin{pmatrix} -7 & 2 \\ 11 & -3 \end{pmatrix}$ (**B**) $\begin{pmatrix} 2 & -7 \\ -3 & 11 \end{pmatrix}$ (**C**) $\begin{pmatrix} -2 & 7 \\ 3 & -11 \end{pmatrix}$ (**D**) $\begin{pmatrix} -3 & 2 \\ 11 & -7 \end{pmatrix}$ (**E**) None of these

Solution:

Answer: D) $\begin{pmatrix} -3 & 2 \\ 11 & -7 \end{pmatrix}$.

27. Simplify

$$\left(\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}}\right)^2$$

as much as possible. Note that the domain of the algebraic expression is $[1, 2]$.

(**A**) 4 (**B**) $4x - 4$ (**C**) $\dfrac{2}{\sqrt{x-1}}$ (**D**) $\dfrac{4}{x - 2\sqrt{x-1}}$ (**E**) None of these

Solution:

Squaring and simplifying, we get

$$
\begin{aligned}
&(\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}})^2 \\
&= 2x + 2\sqrt{x^2 - 4(x-1)} \\
&= 2x + 2\sqrt{(x-2)^2} \\
&= 2x - 2(x-2) \\
&= 4.
\end{aligned}
$$

Answer: A) 4.

28. Let r be a complex number with nonzero imaginary part and let n be the least positive integer such that $r^n = 1$, which of the following is equal to

$$\sum_{k=1}^{n} r^k = r + r^2 + \cdots + r^n?$$

(**A**) nr (**B**) r^{n+1} (**C**) 1 (**D**) 0 (**E**) None of these

Solution:

The sum of all nth roots of unity is equal to 0.

Answer: D) 0**.**

29. If the sum of the squares of all the (complex) roots of the polynomial

$$x^4 - 5x^3 - Cx^2 - 15x + 20$$

is equal to 39, what is the value of C?

(**A**) 5 (**B**) 7 (**C**) 10 (**D**) 13 (**E**) None of these

Solution:

By Vieta's formulas, the sum of squares is equal to $5^2 + 2C = 39$. Therefore, $C = -7$

Answer: B) 7**.**

30. Suppose one bag of marbles contains 3 red marbles and 4 blue marbles, and a second bag contains 5 red marbles and 3 blue marbles. Suppose you choose a bag at random, then draw five marbles without replacement. What is the probability that all five marbles drawn are of the same color?

(**A**) 0 (**B**) $\frac{5}{56}$ (**C**) $\frac{1}{112}$ (**D**) $\frac{5}{144}$ (**E**) None of these

Solution:

Since the probability of drawing 5 marbles of the same color from the first bag is 0, the probability is

$$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} = \frac{1}{112}$$

Answer: C) $\frac{1}{112}$**.**

31. Suppose $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are both arithmetic progressions. If $a_{11} + b_{11} = 100$ and $a_{16} + b_{16} = 120$, determine the value of $a_1 + b_1$.

(**A**) -35 (**B**) 12 (**C**) 60 (**D**) 89 (**E**) None of these

Solution:

Let d_a and d_b be the common differences of sequences $\{a_i\}_i, \{b_i\}_i$, resp., then

$$\begin{aligned} 100 &= a_{11} + b_{11} \\ &= a_1 + 10d_a + b_1 + 10d_b \\ 120 &= a_{16} + b_{16} \\ &= a_1 + 15d_a + b_1 + 15d_b \\ 3 \cdot 100 - 2 \cdot 120 &= a_1 + b_1. \end{aligned}$$

Answer: C) 60**.**

32. Suppose $-2.1 < \log_{18} \frac{1}{432} < -2.099$. Which of these values is the best estimate of $\log_{24} 18$?

(**A**) 0.476 (**B**) 0.667 (**C**) 0.909 (**D**) 1.133 (**E**) None of these

Solution:

$$\begin{aligned}\log_{18}\frac{1}{432} &= -\log_{18} 432\\ &= -\log_{18} 18\cdot 24\\ &= -(1+\log_{18} 24)\\ \log_{18} 24 &\approx 1.1\\ \log_{24} 18 &\approx \frac{1}{1.1}\\ &\approx 0.909\end{aligned}$$

Answer: C) 0.909**.**

33. Define the operation

$$x * y = \big((\sqrt[3]{x}-3)(\sqrt[3]{y}-3)+3\big)^3$$

for all x, y in the interval $(3, \infty)$. Find y, $y > 3$, such that $2026 * y = 2026$.

(**A**) 1 (**B**) 3 (**C**) 27 (**D**) 64 (**E**) None of these

Solution:

$$\begin{aligned}2026 &= 2026 * y\\ &= \Big((\sqrt[3]{2026}-3)(\sqrt[3]{y}-3)+3\Big)^3\\ \sqrt[3]{2026}-3 &= (\sqrt[3]{2026}-3)(\sqrt[3]{y}-3)\\ 1 &= \sqrt[3]{y}-3\\ 4 &= \sqrt[3]{y}\\ y &= 64.\end{aligned}$$

Answer: D) 64**.**

34. Compute the value of the continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}.$$

(You may assume that 1 and 2 alternate indefinitely.)

(**A**) $\dfrac{1+\sqrt{3}}{2}$ (**B**) $\dfrac{1+\sqrt{5}}{2}$ (**C**) $\dfrac{1+\sqrt{7}}{2}$ (**D**) $\dfrac{1+\sqrt{11}}{2}$ (**E**) None of these

Solution:

$$
\begin{aligned}
x &= 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \\
&= 1 + \frac{1}{2 + \frac{1}{x}} \\
&= 1 + \frac{x}{2x+1} \\
&= \frac{3x+1}{2x+1} \\
2x^2 + x &= 3x + 1 \\
2x^2 - 2x - 1 &= 0 \\
x &= \frac{2 \pm \sqrt{12}}{4} \\
x &= \frac{1 \pm \sqrt{3}}{2} \\
x &= \frac{1 + \sqrt{3}}{2}
\end{aligned}
$$

Answer: A) $\dfrac{1+\sqrt{3}}{2}$

35. Suppose a, b, c are integers and consider the following system of two equations in x and y.

$$
\begin{cases} ax + 5y = b \\ 11x + 2y = c \end{cases}
$$

If this system **always** has integer solutions for all possible values of b and c, then what are the possible values of a? (Recall that a solution (x, y) is called an integer solution if both x, y are integers.)

(**A**) $a = 27$ or 28 (**B**) $a = 27$ or 29 (**C**) $a = 28$ or 29 (**D**) $a = 28$ or 30 (**E**) None of these

Solution:

Solving the system results in dividing $2, 5, 11$ by the determinant $2a - 55$, which implies that $2a - 55$ must be a common divisor of $5, 11$. Therefore, the determinant is ± 1, and $a = 27, 28$

Answer: A) $a = 27$ **or** 28

36. Consider the function

$$
f(x) = \frac{x^2 - 1}{x + 1} + \frac{x^3 - 1}{x^2 + x + 1} + \frac{x^4 - 1}{x^3 + x^2 + x + 1} + \cdots + \frac{x^{2026} - 1}{x^{2025} + x^{2024} + \cdots + x + 1}.
$$

Evaluate $f(2)$.

(**A**) 2024 (**B**) 2025 (**C**) 2026 (**D**) 2027 (**E**) None of these

Solution:

$f(x)$ simplifies to $f(x) = (x - 1) + (x - 1) + \cdots + (x - 1) = 2025(x - 1)$ away from roots of unity. Therefore, $f(2) = 2025$.

Answer: B) 2025

37. Find the remainder when you divide

$$x^{20} - 9x^{18} + x^{10} + 3x^9 + x^4 + 2x^3 - x^2 + x$$

by $x + 3$.

(**A**) $x^2 + 3x + 1$ (**B**) $x^{19} + x^9$ (**C**) -36 (**D**) 15 (**E**) None of these

Solution:

Synthetic division or plugging in $x = -3$:

$$(-3)^{20} - 9(-3)^{18} + (-3)^{10} + 3(-3)^9 + (-3)^4 + 2(-3)^3 - (-3)^2 + (-3) = 81 - 54 - 9 - 3$$

Answer: D) 15

38. Compute the sum of the squares of all values of x that make the following expression true (and well-defined).

$$\frac{x^2 - 16}{x^2 - 5x + 6} = \frac{x + 4}{x^2 - 7x + 12}$$

(**A**) 25 (**B**) 29 (**C**) 52 (**D**) 61 (**E**) None of these

Solution:

$$\begin{aligned}
0 &= \frac{x^2 - 16}{x^2 - 5x + 6} - \frac{x + 4}{x^2 - 7x + 12} \\
&= \frac{(x^2 - 16)(x - 4)}{(x - 2)(x - 3)(x - 4)} - \frac{(x + 4)(x - 2)}{(x - 2)(x - 3)(x - 4)} \\
&= \frac{(x^2 - 16)(x - 4) - (x + 4)(x - 2)}{(x - 2)(x - 3)(x - 4)} \\
&= \frac{(x + 4)((x - 4)^2 - (x - 2))}{(x - 2)(x - 3)(x - 4)} \\
&= \frac{(x + 4)(x^2 - 9x + 18)}{(x - 2)(x - 3)(x - 4)} \\
&= \frac{(x + 4)(x - 6)(x - 3)}{(x - 2)(x - 3)(x - 4)} \\
(-4)^2 + 6^2 &= 52
\end{aligned}$$

Answer: C) 52

39. Write the following complex number in the form $a + bi$, where a, b are real numbers.

$$\frac{5}{3 + 4i} + \frac{13}{5 + 12i} + \frac{5}{3 - 4i} - \frac{13}{5 - 12i}$$

(**A**) $\frac{3}{5} - \frac{12}{13}i$ (**B**) $\frac{3}{5} + \frac{12}{13}i$ (**C**) $\frac{6}{5} - \frac{24}{13}i$ (**D**) $\frac{6}{5} + \frac{24}{13}i$ (**E**) None of these

Solution:

$$
\begin{aligned}
&\frac{5}{3+4i}+\frac{13}{5+12i}+\frac{5}{3-4i}-\frac{13}{5-12i}\\
&=\frac{5(3-4i)}{25}+\frac{5(3+4i)}{25}+\frac{13(5-12i)}{169}-\frac{13(5+12i)}{169}\\
&=\frac{6}{5}+\frac{-24i}{13}
\end{aligned}
$$

Answer: C) $\frac{6}{5}-\frac{24}{13}i$

40. Let x be a real number such that $x^3-\frac{1}{x^3}=14$. Find the value of $x-\frac{1}{x}$.

(**A**) -3 (**B**) -1 (**C**) 1 (**D**) 3 (**E**) None of these

Solution:

Let $t=x-\frac{1}{x}$.

$$
\begin{aligned}
t^3 &= x^3-3(x-1/x)-1/x^3\\
&=14-3t\\
0 &= t^3+3t-14\\
&=(t-2)(t^2+2t+7)
\end{aligned}
$$

Since this polynomial only has a single real root of 2, $t=2$.

Answer: E) None of these

41. Consider the system of inequalities

$$
\begin{cases}
3x-2y & \leq 4,\\
x+4y & \geq 1,\\
5x+3y & \leq 12
\end{cases}
$$

If (x,y) is a solution to the system. What is the maximum possible value of x?

(**A**) $\frac{36}{19}$ (**B**) $\frac{45}{17}$ (**C**) $\frac{63}{19}$ (**D**) $\frac{72}{23}$ (**E**) None of these

Solution:

The maximum value occurs at the intersection of $3x-2y=4$ and $5x+3y=12$:

$$
\begin{aligned}
3x-2y=4 &\implies y=\frac{3x-4}{2}\\
5x+3y&=12\\
5x+3\cdot\frac{3x-4}{2}&=12\\
10x+9x-12&=24\\
19x&=36\\
x&=\frac{36}{19}
\end{aligned}
$$

Answer: A) $\frac{36}{19}$

42. Compute the value of

$$2026^2 - 2025^2 + 2024^2 - 2023^2 + \cdots + 2002^2 - 2001^2.$$

(**A**) 48348 (**B**) 52351 (**C**) 104026 (**D**) 123721 (**E**) None of these

Solution:

$$\begin{aligned}
&= 2026^2 - 2025^2 + 2024^2 - 2023^2 + \cdots + 2002^2 - 2001^2 \\
&= 4051 + 4051 - 4 + \cdots + 4051 - 48 \\
&= 13 \cdot 4051 - 4 \cdot \frac{12 \cdot 13}{2} \\
&= 52663 - 312 \\
&= 52351
\end{aligned}$$

Answer: B) 52351

43. Consider the function $f(x) = 2 + ax + 10x^2$. Which of the following cannot be a zero of $f(x)$ for any **integer** value of a?

(**A**) $-\frac{2}{5}$ (**B**) $-\frac{1}{2}$ (**C**) $\frac{1}{5}$ (**D**) $\frac{5}{2}$ (**E**) None of these

Solution:

$$\begin{aligned}
&x = -2/5 \implies 2 - 2a/5 + 8/5 = 0 \text{ possible, e.g. } a = 1 + 4 \\
&x = -1/2 \implies 2 - a/2 + 5/2 = 0 \text{ possible, e.g. } a = 4 + 5 \\
&x = 1/5 \implies 2 + a/5 + 2/5 = 0 \text{ possible, e.g. } a = -(10 + 2) \\
&x = 5/2 \implies 2 + 5a/2 + 125/2 = 0 \text{ impossible}
\end{aligned}$$

Answer: D) $\frac{5}{2}$

44. How many pairs of integers (x, y) satisfy

$$\sqrt{xy} = \sqrt{x+y} + \sqrt{x} + \sqrt{y} \ \ ?$$

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) Infinitely many (**E**) None of these

Solution:

If $x = 0$, then $y = 0$ is the only solution and vice-versa. Assume $x, y \neq 0$.

$$\begin{aligned}
&\sqrt{xy} = \sqrt{x+y} + \sqrt{x} + \sqrt{y} \\
&\iff \sqrt{xy} - (\sqrt{x} + \sqrt{y}) = \sqrt{x+y} \\
&\implies (\sqrt{xy} - (\sqrt{x} + \sqrt{y}))^2 = x + y \\
&\iff xy + x + y - 2(y\sqrt{x} + x\sqrt{y} - \sqrt{xy})) = x + y \\
&\iff \sqrt{xy}(\sqrt{xy} - 2\sqrt{x} - 2\sqrt{y} + 2) = 0 \\
&\iff (2 - \sqrt{x})(2 - \sqrt{y}) = 2
\end{aligned}$$

Hence, $2-\sqrt{x} = \pm 1, \pm 2$ implies $x = 1, 9, 16, 0$ and $y = 0, 16, 9, 1$, resp. Since $x, y \neq 0$, this gives us two pairs $(9, 16), (16, 9)$ in addition to the original pair $(0, 0)$.

Answer: C) 3

45. Find the 2026th term of the unique non-decreasing sequence of positive integers starting at 1 such that n occurs exactly n times (for all positive integers n):

$$\underbrace{1}_{\text{1 copy}}, \underbrace{2,2}_{\text{2 copies}}, \underbrace{3,3,3}_{\text{3 copies}}, \underbrace{4,4,4,4}_{\text{4 copies}}, \ldots$$

(**A**) 64 (**B**) 73 (**C**) 144 (**D**) 2026 (**E**) None of these

Solution:

The first occurrence of n is at term $t(n) = 1 + \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n + 2)$. Note that $t(n+1) = t(n) + n$. Trying out some values (note that n should be close to $\sqrt{2 \cdot 2026}$ which is not far from 65):

$$\begin{aligned}
t(100) &= 1 + 50 \cdot 99 > 2026 \\
t(50) &= 1 + 25 \cdot 49 = 1 + (1250 - 25) < 2026 \\
t(60) &= 1 + 30 \cdot 59 = 1 + (1800 - 30) < 2026 \\
t(70) &= 1 + 35 \cdot 69 = 1 + (2450 - 35) > 2026 \\
t(65) &= 1 + 65 \cdot = 32 = 1 + 2080 > 2026 \\
t(64) &= 2081 - 64 = 2017 < 2026
\end{aligned}$$

Therefore, the 2026th term lies between the first occurrence of 64 and the first occurrence of 65. Therefore, it is equal to 64.

Answer: A) 64

46. For any real number x, recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x (this is called the floor function). Find the smallest nonnegative value of x such that

$$\lfloor x \rfloor + 2\lfloor x \rfloor + 3\lfloor x \rfloor \neq \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor.$$

(**A**) 0 (**B**) $\frac{1}{6}$ (**C**) $\frac{1}{3}$ (**D**) $\frac{1}{2}$ (**E**) None of these

Solution:

If $0 < x < \frac{1}{3}$, then $0 < 2x < \frac{2}{3}$ and $0 < 3x < 1$ implies that $\lfloor x \rfloor = \lfloor 2x \rfloor = \lfloor 3x \rfloor = 0$. Therefore, $0 = \lfloor x \rfloor + 2\lfloor x \rfloor + 3\lfloor x \rfloor = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$ holds whenever $0 < x < \frac{1}{3}$. When $x = 1/3$, $\lfloor 3x \rfloor = 1$ and the other terms are all zero.

Answer: C) $\frac{1}{3}$

47. Recall that, for any nonnegative integers n and r, $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. If $\binom{n}{4} = 330$, determine the value of n.

(**A**) 7 (**B**) 11 (**C**) 13 (**D**) 31 (**E**) None of these

Solution:

Either check of the options or notice that 330 has a factor of 11 and check 11.

Answer: B) 11

48. Consider the polynomial function

$$f(x) = x^4 + ax^3 + bx^2 + cx + d.$$

If for any positive integer k, $1 \le k \le 4$,

$$\underbrace{f(f(\cdots f(}_{k \text{ times}}0)\cdots)) = k^2,$$

Compute the value of $a + b + c + d$.

(**A**) 3 (**B**) -5 (**C**) 6 (**D**) -18 (**E**) None of these

Solution:

Note that $a+b+c+d = f(1)-1$ and that $f(0) = 1^2 = 1$, then $f(1) = f(f(0)) = 2^2$. Therefore, $a+b+c+d = 3$.

Answer: A) 3

49. Consider the functions

$$f(x) = \frac{1}{\sqrt{1+x^2}}, \quad g(x) = \sqrt{1-x}, \quad h(x) = (x+2026)^{1/3}.$$

Which of the following composite functions have their domain equal to the set of **all real numbers**?

(**A**) $f \circ g \circ h$ (**B**) $g \circ h \circ f$ (**C**) $g \circ f \circ h$ (**D**) $f \circ h \circ g$ (**E**) None of these

Solution:

$h(0)$ is greater than 1 so $g(h(0))$ is not defined. Similarly, $h(f(0)) > 1$ makes $g(h(f(0))$ undefined. $f \circ h \circ g$ is undefined for all $x > 1$. Since $f(x) \le 1$ for all real numbers, $g(f(x))$ is always defined, and so is $g(f(h(x)))$. Therefore, the domain of $g \circ f \circ h$ is the set of all real numbers.

Answer: C) $g \circ f \circ h$

50. Anne, Bess, and Carla give \$$a$, \$$b$, and \$$c$, respectively, when they donate to a particular charity. The first month, Carla forgot to donate, but the other two remembered. Similarly, in the second month, Bess forgot to donate, but the other two remembered. Finally, in the third month, Anne forgot to donate, but the other two remembered. If the charity received a total of \$120, \$200, and \$280 for the first, second, and third months, resp., from Anne, Bess, and Carla, what is the value of a?

(**A**) \$40 (**B**) \$80 (**C**) \$160 (**D**) \$200 (**E**) None of these

Solution:

Anne is a little stingy: $a = 20, b = 100, c = 180$.

Answer: E) None of these