The 45th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: February 28, 2026 at Regional Testing Sites

Second Round (by invitation): April 18, 2026 at Auburn University Montgomery

COMPREHENSIVE EXAMINATION

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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions are not arranged in order of difficulty. For each question, choose the best of the five options labeled A, B, C, D and E. Calculators are NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a paper with: all questions answered correctly earns a score of 250, all questions left blank earns a score of 50, and all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Analysis
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, Northrop Grumman, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Two complementary angles have measures $2x$ and $3x - 10$. Find the supplement of the larger angle.

(**A**) $130°$ (**B**) $135°$ (**C**) $140°$ (**D**) $145°$ (**E**) None of these

Solution:

Since the two angles are complementary,

$$2x + 3x - 10 = 90.$$

Then

$$5x - 10 = 90 \quad \Rightarrow \quad x = 20.$$

So the smaller angle is

$$2x = 40°,$$

and the larger angle is

$$3x - 10 = 50°.$$

The supplement of the larger angle is

$$180° - 50° = 130°.$$

Answer: A) $130°$

2. If $\sin^4 x + \cos^4 x = \dfrac{21}{41}$ and $x \in \left[-\dfrac{\pi}{4}, 0\right]$, compute $\cos(2x)$.

(**A**) $\dfrac{1}{\sqrt{41}}$ (**B**) $\dfrac{2}{\sqrt{41}}$ (**C**) $\dfrac{3}{\sqrt{41}}$ (**D**) $\dfrac{4}{\sqrt{41}}$ (**E**) None of these

Solution: By squaring the Pythagorean identity, we get $2\sin^2 x \cos^2 x = \dfrac{20}{41}$. Thus, $\sin^2(2x) = \dfrac{40}{41}$. This gives $\cos^2(2x) = \dfrac{1}{41}$. Since $\cos(2x) > 0$, we get $\cos(2x) = \dfrac{1}{\sqrt{41}}$.

Answer: A) $\dfrac{1}{\sqrt{41}}$

3. Alice has \$2.00 in quarters and dimes. Bob has \$1.60 in nickels and dimes. Suppose Alice and Bob have the same number of dimes and the same total number of coins. How many dimes do they have together?

(**A**) 18 (**B**) 22 (**C**) 26 (**D**) 30 (**E**) None of these

Solution: Let Q be the number of quarters and D the number of dimes that Alice has. The condition tells us that Bob also has D dimes and Q nickels. This sets up the system of equations

$$\begin{cases} 25Q + 10D = 200 \\ 5Q + 10D = 160. \end{cases}$$

We solve this to get $Q = 2$, $D = 15$.

Answer:D) 30

4. Which of the following cannot be the last two digits of a positive whole number raised to the power 2026?

(**A**) 14 (**B**) 29 (**C**) 64 (**D**) 89 (**E**) None of these

Solution: Note $x^{20206} \pmod 4 \in \{0, 1\}$. Thus, A) is impossible. It is easy to check the other are possible by considering modulo 25.

Answer: A) 14

5. Determine the minimum value of the function

$$f(x) = 10 - 3\cos(3x) - \frac{2\sin(6x)}{\cos(3x)},$$

on the domain where it is defined.

(**A**) 4 (**B**) 5 (**C**) 6 (**D**) 7 (**E**) None of these

Solution: We apply the double angle identity to get

$$f(x) = 10 - 3\cos(3x) - 4\sin(3x).$$

Thus, we get the minimum is 5 by writing the combination $3\cos(3x) + 4\sin(3x)$ as a phase shift, i.e.,

$$3\cos(3x) + 4\sin(3x) = \sqrt{3^2 + 4^2}\cos(\theta - \phi)$$

where $\phi = \tan^{-1}(4/3)$. Since the max of the above expression is 5, the min of f is 5. **Answer: B)** 5

6. Let $x = \frac{p}{q}$ be the positive reduced fraction with the smallest possible q such that $p + q \geq 10$ and

$$\left|\sqrt{2} - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Determine $p + q$.

(**A**) 10 (**B**) 11 (**C**) 12 (**D**) 13 (**E**) None of these

Solution: Note that $x = \frac{7}{5}$ works. We need to check this is the smallest q. Indeed, no other $\frac{p}{5}$ works. The only other possibility could be $\frac{p}{4}$; however, then p must be odd and neither $\frac{5}{4}$ nor $\frac{7}{4}$ works.

Answer: C) 12

7. An ant is trapped on the lattice grid with corners at $(0, 0), (3, 0), (3, 3)$, and $(0, 3)$ and can only move up or to the right. Suppose the ant begins at $(0, 0)$ and walks on this grid until reaching a point (n, m) with either $n = 3$ or $m = 3$, where then the ant freezes. If at each point, the ant has a $\frac{1}{3}$ chance of moving up and a $\frac{2}{3}$ chance of moving to the right, what is the probability the ant passes through the point $(2, 2)$?

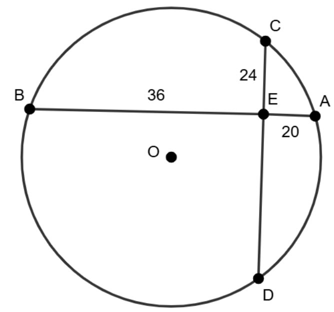
(**A**) $\frac{5}{27}$ (**B**) $\frac{2}{9}$ (**C**) $\frac{7}{27}$ (**D**) $\frac{8}{27}$ (**E**) None of these

Solution: The probability of doing a given path is completely determined by the number of up moves and right moves it consists of. Thus, this is equivalent to counting the number of ways of ordering two up moves and two right moves, each with probability $\frac{4}{81}$. There are $\binom{4}{2}$ such paths. Thus, $\frac{24}{81} = \frac{8}{27}$.

Answer: D) $\frac{8}{27}$

8. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at right angles at E. If $AE = 20$, $EB = 36$, and $CE = 24$, determine the measure of DE.

(**A**) 30 (**B**) 38 (**C**) 42 (**D**) 46 (**E**) None of these



Solution: In the figure shown, we see that

$$EB \cdot AE = CE \cdot DE.$$

Let $x = DE$. Then

$$36 \cdot 20 = 24 \cdot x$$
$$720 = 24x$$
$$x = 30.$$

Therefore,

$$DE = 30.$$

Answer: A) 30

9. A rectangle $ABCD$ is such that AB is tangent to circle ω with C and D on ω. If ω has radius 1 and $AD = y$, find the area of $ABCD$ in terms of y.

(**A**) $2y^{1/2}(2-y)^{3/2}$ (**B**) $2y^{3/2}(2-y)^{1/2}$

(**C**) $4y^{1/2}(2-y)^{3/2}$ (**D**) $4y^{3/2}(2-y)^{1/2}$ (**E**) None of these

Solution: We apply the Pythagorean theorem to get $AB = 2\sqrt{2y - y^2}$.

Answer: B) $2y^{\frac{3}{2}}(2-y)^{\frac{1}{2}}$

10. Let $f(n)$ be a function defined on the integers such that $f(n) = 1$ if n is a multiple of 3 and $f(n) = 0$ otherwise. Let $N = \sum_{n=1}^{2026}(-1)^{f(n)}n$. Determine the remainder of N when divided by 5.

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

Solution: We have

$$N = 2026 + \sum_{k=0}^{674} 3k = 2026 + \frac{3 \cdot 675 \cdot 674}{2} \equiv 1.$$

Answer: A) 1

11. Find a solution of the equation $\frac{3^{x-2}}{2} = 3^{-1} + 3^{-(x+2)}$.

(**A**) $\log_3(3 - \sqrt{11})$ (**B**) $\log_3(3 - \sqrt{7})$ (**C**) $\log_3(3 + \sqrt{7})$ (**D**) $\log_3(3 + \sqrt{11})$ (**E**) None of these

Solution:

$$
\begin{aligned}
\frac{3^{x-2}}{2} &= \frac{1}{3} + 3^{-(x+2)} \\
3^{x-2} &= \frac{2}{3} + 2 \cdot 3^{-(x+2)} \\
3^{2x} &= \frac{2}{3} \cdot 3^{x+2} + 2 \\
3^{2x} &= 6 \cdot 3^{x} + 2 \\
3^{2x} - 6 \cdot 3^{x} - 2 &= 0 \\
3^{x} &= \frac{6 \pm \sqrt{36+8}}{2} \\
3^{x} &= 3 + \sqrt{11} \text{(discard negative solution)} \\
x &= \log_3(3+\sqrt{11})
\end{aligned}
$$

Answer: D) $\log_3(3+\sqrt{11})$**.**

12. At a carnival game, there is a bowl of balls labeled 1 through 100. It costs a contestant N dollars to participate in the game. During the game, a contestant simultaneously selects 3 balls randomly. The contestant wins 2 dollars for each even ball and wins nothing for each odd ball. What is the least entry cost N so that the expected net gain of the contestant is negative?

(**A**) \$1 (**B**) \$2 (**C**) \$3 (**D**) \$4 (**E**) None of these

Solution: The expected gain is

$$6 \cdot \frac{\binom{50}{3}}{\binom{100}{3}} + 4 \cdot \frac{\binom{50}{2}\binom{50}{1}}{\binom{100}{3}} + 2 \cdot \frac{\binom{50}{1}\binom{50}{2}}{\binom{100}{3}} = 3.$$

Thus, $N > 3$.

Answer: D) \$4

13. The equation $4x + 5y + 6z = 7$ has infinitely many **integer** solutions (solutions such that x, y, z are all integers). If x, y, z is a solution and x is a multiple of 3, which of the following is a possible value of y for some integer value of z?

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) None of these

Solution: If $y = 2, 4$, then the left side of the equation is even which is impossible because the right side is odd. If $y = 3$, then left side of the equation is divisible by 3 which is impossible because the right side is not. If $y = 5$, then a possible solution is $x = 0, y = 5, z = -3$.

Answer: D) 5

14. A right circular cone has a base area of 25π, a lateral surface area of 65π, and a total surface area of 90π. What is its volume?

(**A**) 80π (**B**) 90π (**C**) 100π (**D**) 110π (**E**) None of these

Solution: For a right circular cone, the base area is πr^2, so

$$\pi r^2 = 25\pi \Rightarrow r = 5.$$

The lateral surface area is $r\pi s = 65\pi$, so

$$5\pi s = 65\pi \Rightarrow s = 13.$$

Now

$$r^2 + h^2 = s^2,$$
$$25 + h^2 = 169 \Rightarrow h^2 = 144 \Rightarrow h = 12.$$

The volume is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}(25\pi)(12) = 100\pi.$$

Answer: C) 100π

15. Suppose $\sin(\theta) < 0$ and $\cos(2\theta) > 0$. Which of the following could be a possible angle θ?

(**A**) $\frac{2\pi}{3}$ (**B**) $\frac{5\pi}{6}$ (**C**) $\frac{7\pi}{6}$ (**D**) $\frac{4\pi}{3}$ (**E**) None of these

Solution: We just check each choice.

Answer: C) $\frac{7\pi}{6}$

16. Suppose the complex number $z = a + bi$ satisfies $(a+2bi)^2 = z(a - bi)$. Which statement is necessarily true?

(**A**) z is real (**B**) z is purely imaginary (**C**) z has modulus 1 (**D**) $z = 0$ (**E**) None of these

Solution: We get $(a^2 - 4b^2) + (4ab)i = a^2 + b^2$. Matching real and imaginary parts, $b = 0$ and the equation is satisfied for any a.

Answer: A) z **is real.**

17. The equation $\left|\frac{3x-2y}{3}\right| + \left|\frac{y}{4}\right| = 1$ defines a quadrilateral in the coordinate plane. Find its area.

(**A**) 4 (**B**) 6 (**C**) 8 (**D**) 10 (**E**) None of these

Solution: This is a parallellogram. We compute the area of the two triangles above and below the x-axis. Each has a base of 2 and height 4.

Answer: C) 8

18. Simplify the following:

$$(x^2 - 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1).$$

(**A**) x^{10} (**B**) $x^{10} - 1$ (**C**) $x^{10} + x^8 + x^6 + x^4 + x^2 + 1$

(**D**) $x^{10} - x^8 + x^6 - x^4 + x^2 - 1$ (**E**) None of these

Solution:

The inequality asserts that m_1, m_3 are negative and m_2, m_4 are positive. By the first and second equations,

$$\begin{aligned}
&(x^2 - 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1)\\
&= (x - 1)(x^4 + x^3 + x^2 + x + 1)\cdot\\
&\quad (x + 1)(x^4 - x^3 + x^2 - x + 1)\\
&= (x^5 - 1)(x^5 + 1)\\
&= x^{10} - 1
\end{aligned}$$

Answer: B) $x^{10} - 1$**.**

19. Which of the following polynomials $f(x)$ satisfies $f(3+x) = f(3-x)$ for all x?

(**A**) $f(x) = x^2 - 9$ (**B**) $f(x) = x^2 + 6x + 9$ (**C**) $f(x) = (x^2 - 6x + 9)^3$

(**D**) $f(x) = (x^3 + 3x^2 + 6x + 9)^2$ (**E**) None of these

Solution:

This property is symmetry about the line $x = 3$. We can check specific values, like $x = 3$, for which (A) gives the false statement $-9 = 36 - 9$, (B) gives the false statement $9 = 36 + 36 + 9$, (C) gives the true statement $9^3 = 9^3$, and (D) gives the false statement $9^2 = (6^3 + 3 \cdot 36 + 36 + 9)^2$.

Answer: C) $f(x) = (x^2 - 6x + 9)^3$**.**

20. Alice has 4 piggy banks. In total, the 4 banks have 99 pennies, 17 nickels, and 5 dimes. She writes down the amount stored in each bank. Which could be the largest number she wrote down?

(**A**) \$0.54 (**B**) \$0.56 (**C**) \$0.58 (**D**) \$0.60 (**E**) None of these

Solution: There is $99 + 85 + 50 = 234$ cents total. By pigeonhole, one must have over $234/4 = 58.5$ cents.

Answer: D) \$0.60

21. The area of a circle is eight times the circumference. Find the area of this circle.

(**A**) 144π (**B**) 196π (**C**) 225π (**D**) 256π (**E**) None of these

Solution: Let r be the radius. The area is πr^2 and the circumference is $2\pi r$. Given:

$$\pi r^2 = 8(2\pi r).$$

$$\pi r^2 = 16\pi r \Rightarrow r(r-16) = 0.$$

Thus $r = 16$. Then the area is

$$\pi r^2 = 256\pi.$$

Answer: D) 256π

22. In a bag, there is one fair coin and two identical weighted coins. Suppose a person conducts an experiment where one of the coins is randomly selected from the bag and then flipped twice. If the probability that two heads occur is $\frac{19}{100}$, what is the probability that exactly zero heads occurred?

(**A**) $\frac{93}{300}$ (**B**) $\frac{97}{300}$ (**C**) $\frac{101}{300}$ (**D**) $\frac{103}{300}$ (**E**) None of these

Solution: Note (via a tree diagram) we have

$$\frac{19}{100} = \frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3}p^2.$$

We solve to get $p = \frac{2}{5}$. Then, we compute $\frac{1}{12} + \frac{2}{3} \cdot \frac{9}{25} = \frac{97}{300}$.

Answer: B) $\frac{97}{300}$

23. Let $f(n) = 1! + 2! + \cdots n!$ for any integer $n \geq 1$. What is the remainder of $f(100)$ when divided by 6?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution:

Since $n!$ is divisible by 6 for all $n \geq 3$, we only need to consider $f(2) = 1! + 2! = 3$. Therefore, $f(100)$ divided by 6 has remainder 3.

Answer: C) 3**.**

24. Suppose p and q are prime numbers such that $3p + q = 2026$, which of these is possible value of q?

(**A**) 3 (**B**) 23 (**C**) 43 (**D**) 83 (**E**) None of these

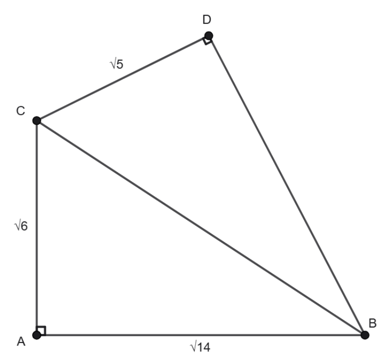
Solution:

q has the same remainder as 2026 when divided by 3. Therefore, the only possible solutions are 43 or E. The number $2026 - 43 = 1983 = 3 \cdot 661$ is three times a prime.

Answer: C) 43**.**

25. In the given figure, find the length of $\overline{BD}$. (The angles at A and D are right angles and the lengths are $AB = \sqrt{14}$, $AC = \sqrt{6}$, and $CD = \sqrt{5}$.)

(**A**) $\sqrt{10}$ (**B**) $\sqrt{15}$ (**C**) $2\sqrt{10}$ (**D**) $4\sqrt{5}$ (**E**) None of these



Solution: Using the Pythagorean theorem in $\triangle ABC$,

$$AC^2 + AB^2 = BC^2,$$

$$16^2 + 114^2 = BC^2 \Rightarrow BC = 2\sqrt{15}.$$

Again in $\triangle BCD$,

$$CD^2 + BD^2 = BC^2,$$

$$15^2 + BD^2 = (2\sqrt{15})^2 = 60,$$

$$BD^2 = 15 \Rightarrow BD = \sqrt{15}.$$

Answer: B) $\sqrt{15}$

26. Let C_1 and C_2 be concentric circles with C_1 being the larger. A chord of C_1 tangent to C_2 has length 28. What is the area of the region which lies between C_1 and C_2?

(**A**) 121π (**B**) 144π (**C**) 196π (**D**) 225π (**E**) None of these

Solution: Let the radii be R and r. Because the chord is tangent to C_2, the distance from the center to the chord is r, so

$$R^2 - r^2 = 14^2 = 196.$$

Thus the required area is

$$\pi(R^2 - r^2) = 196\pi.$$

Answer: C) 196π

27. An equilateral triangle is inscribed in a circle with radius 2. Find the perimeter of the equilateral triangle.

(**A**) $4\sqrt{3}$ (**B**) $6\sqrt{3}$ (**C**) $8\sqrt{3}$ (**D**) $10\sqrt{3}$ (**E**) None of these

Solution: The central angle is $120°$ and half-angle is $60°$. Using $\cos 30° = \dfrac{\sqrt{3}}{2}$ gives half-side

$$\frac{s}{2} = \frac{\sqrt{3}}{2} \cdot 2 = \sqrt{3}.$$

Thus $s = 2\sqrt{3}$ and the perimeter is

$$3s = 6\sqrt{3}.$$

Answer: B) $6\sqrt{3}$

28. Suppose $x = \tan\left(\dfrac{\theta}{2}\right)$ for some $-1 < x < 1$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Compute $\sec(\theta)$ in terms of x.

(**A**) $\dfrac{1-x^2}{1+x^2}$ (**B**) $\dfrac{-2x}{1+x^2}$ (**C**) $\dfrac{1+2x-x^2}{1-x^2}$ (**D**) $\dfrac{1+x^2}{1-x^2}$ (**E**) None of these

Solution: We apply the tangent double angle formula and then the Pythagorean identity.

Answer: D) $\dfrac{1+x^2}{1-x^2}$

29. Compute $\cos(70°)\cos(55°) + \sin(70°)\sin(55°)$.

(**A**) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ (**B**) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ (**C**) $\dfrac{\sqrt{5}-1}{4}$ (**D**) $\dfrac{\sqrt{5}+1}{4}$ (**E**) None of these

Solution: Apply the cosine angle-sum formula to get $\cos(70-55) = \cos(15) = \cos(45)\cos(30)+\sin(45)\sin(30) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Answer: B) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

30. Find the value of

$$\sum_{n=1}^{99} \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

(**A**) $3\sqrt{5}$ (**B**) 9 (**C**) $10\sqrt{2}$ (**D**) 26 (**E**) None of these

Solution:

$$= \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots \frac{1}{\sqrt{99}+\sqrt{100}}$$
$$= \frac{\sqrt{2}-1}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \cdots \frac{\sqrt{100}-\sqrt{99}}{100-99}$$
$$10 - 1 = 9.$$

Answer: B) 9**.**

31. Which of the following is the multiplicative inverse of $\begin{pmatrix} 7 & 2 \\ 11 & 3 \end{pmatrix}$?

(**A**) $\begin{pmatrix} -7 & 2 \\ 11 & -3 \end{pmatrix}$ (**B**) $\begin{pmatrix} 2 & -7 \\ -3 & 11 \end{pmatrix}$ (**C**) $\begin{pmatrix} -2 & 7 \\ 3 & -11 \end{pmatrix}$ (**D**) $\begin{pmatrix} -3 & 2 \\ 11 & -7 \end{pmatrix}$ (**E**) None of these

Solution:

Answer: D) $\begin{pmatrix} -3 & 2 \\ 11 & -7 \end{pmatrix}$**.**

32. The line $x + y = k$ is tangent to the circle $x^2 + y^2 = 1$. Find the value of k.

(**A**) 1 (**B**) $\frac{3\pi}{4}$ (**C**) $\sqrt{2}$ (**D**) $\frac{\sqrt{2}}{2}$ (**E**) None of these

Solution: A tangent has distance from center equal to radius:

$$\frac{|k|}{\sqrt{1^2 + 1^2}} = 1.$$

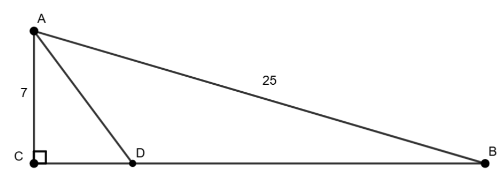
Thus

$$\frac{|k|}{\sqrt{2}} = 1 \quad \Rightarrow \quad |k| = \sqrt{2}.$$

Positive value gives $k = \sqrt{2}$.

Answer: C) $\sqrt{2}$

33. In $\triangle BCA$, angle C is a right angle and $\overline{AD}$ is a bisector of angle $\angle BAC$. Given that $AC = 7$ and $AB = 25$, find the length of segment $\overline{CD}$.



(**A**) $\frac{21}{4}$ (**B**) $\frac{24}{7}$ (**C**) 12

(**D**) $\frac{75}{7}$ (**E**) None of these

Solution: By the angle bisector theorem and area decomposition,

$$CD = \frac{21}{4}.$$

Answer: A) $\frac{21}{4}$

34. Define the operation

$$x * y = \left((\sqrt[3]{x} - 3)(\sqrt[3]{y} - 3) + 3\right)^3$$

for all x, y in the interval $(3, \infty)$. Find y, $y > 3$, such that $2026 * y = 2026$.

(**A**) 1 (**B**) 3 (**C**) 27 (**D**) 64 (**E**) None of these

Solution:

$$
\begin{aligned}
2026 &= 2026 * y \\
&= \left((\sqrt[3]{2026}-3)(\sqrt[3]{y}-3)+3\right)^3 \\
\sqrt[3]{2026}-3 &= (\sqrt[3]{2026}-3)(\sqrt[3]{y}-3) \\
1 &= \sqrt[3]{y}-3 \\
4 &= \sqrt[3]{y} \\
y &= 64.
\end{aligned}
$$

Answer: D) 64**.**

35. Let (h,k) denote the center and r the radius of the circle given by

$$x^2+2x+y^2-4y=4.$$

What is the sum $h^2+k^2+r^2$?

(**A**) 8 (**B**) 10 (**C**) 12 (**D**) 14 (**E**) None of these

Solution: Completing the square:

$$(x+1)^2+(y-2)^2=3^2.$$

Thus $h=-1$, $k=2$, $r=3$. So

$$h^2+k^2+r^2=1+4+9=14.$$

Answer: D) 14

36. Suppose x and y are consecutive integers such that $x^3-y^3=721$ and $x^2-y^2=31$. Find the product xy.

(**A**) 160 (**B**) 200 (**C**) 225 (**D**) 240 (**E**) None of these

Solution:

Using the factorizations of $x^3-y^3=(x-y)(x^2+xy+y^2)$ and $x^2-y^2=(x-y)(x+y)$, note that $x-y=1$ and $(x+y)^2-(x^2+xy+y^2)=xy=31^2-721=961-721=240$. **Answer: D)** 240**.**

37. What is the smallest x value of the intersection points of the graphs of $f(x)=10-|x-1|$ and $g(x)=|x+2|$.

(**A**) $-\dfrac{11}{2}$ (**B**) $-\dfrac{7}{2}$ (**C**) $\dfrac{7}{2}$ (**D**) $\dfrac{11}{2}$ (**E**) None of these

Solution:

The left-most parts of the functions ($x<1$ for $f(x)$, $x<-2$ for $g(x)$) intersect at

$$
\begin{aligned}
10+x-1 &= -x-2 \\
2x &= -11 \\
x &= -11/2.
\end{aligned}
$$

Answer: A) $-\dfrac{11}{2}$**.**

38. For an integer $a > 1$, define the function $f_a(x) = \tan(\log_a(x))$. Suppose $2^{27\pi/2}$ is not in the domain of f_a. How many possible integer values are there for a?

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) None of these

Solution: We have that $a^{\pi(\frac{1}{2}+k)} = 2^{27\pi/2}$ which gives $a^{2k+1} = 2^{27}$. We just want to compute the number of odd factors of 27.

Answer: C) 4

39. Let g be the function defined on the natural numbers where $g(n)$ is the number of ones in the binary representation of n. Define $f(n) = g(n+1) - g(n)$. How many $1 \le n \le 100$ are such that $f(n) < 0$?

(**A**) 22 (**B**) 23 (**C**) 24 (**D**) 25 (**E**) None of these

Solution: We recursively count this. Let S_M be the number of $0 \le n \le M-1$ such that $f(n) < 0$. If $M \ge 2^K$, then

$$S_M = S_{2^K} + S_{M-2^K}.$$

Moreover, $S^{2^K} = 2S_{2^{K-1}}$. This gives

$$S_{100} = S_{64} + S_{32} + S_4 = 16 + 8 + 1 = 25.$$

Answer: D) 25

40. Simplify

$$\left(\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}}\right)^2$$

as much as possible. Note that the domain of the algebraic expression is $[1, 2]$.

(**A**) 4 (**B**) $4x - 4$ (**C**) $\frac{2}{\sqrt{x-1}}$ (**D**) $\frac{4}{x-2\sqrt{x-1}}$ (**E**) None of these

Solution:

Squaring and simplifying, we get

$$\begin{aligned}
&(\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}})^2 \\
&= 2x + 2\sqrt{x^2 - 4(x-1)} \\
&= 2x + 2\sqrt{(x-2)^2} \\
&= 2x - 2(x-2) \\
&= 4.
\end{aligned}$$

Answer: A) 4.

41. Let r be a complex number with nonzero imaginary part and let n be the least positive integer such that $r^n = 1$, which of the following is equal to

$$\sum_{k=1}^{n} r^k = r + r^2 + \cdots + r^n?$$

(**A**) nr (**B**) r^{n+1} (**C**) 1 (**D**) 0 (**E**) None of these

Solution:

The sum of all nth roots of unity is equal to 0.

Answer: D) 0**.**

42. If the sum of the squares of all the (complex) roots of the polynomial

$$x^4 - 5x^3 - Cx^2 - 15x + 20$$

is equal to 39, what is the value of C?

(**A**) 5 (**B**) 7 (**C**) 10 (**D**) 13 (**E**) None of these

Solution:

By Vieta's formulas, the sum of squares is equal to $5^2 + 2C = 39$. Therefore, $C = 7$

Answer: B) 7**.**

43. Find the perimeter of the triangle whose vertices are $(1, -1),\ (5, -1),\ (3, 8)$.

(**A**) $\sqrt{85} + 2$ (**B**) $2\sqrt{21} + 4$ (**C**) $\sqrt{63} + 2$ (**D**) $2\sqrt{85} + 4$ (**E**) None of these

Solution:

$$D_{AB} = 4, \quad D_{AC} = \sqrt{85}, \quad D_{BC} = \sqrt{85}.$$

Thus, the perimeter is

$$4 + \sqrt{85} + \sqrt{85} = 2\sqrt{85} + 4.$$

Answer: D) $2\sqrt{85} + 4$

44. Two similar polygons have corresponding sides 16 and 24. If the area of the larger polygon is 180, find the area of the smaller polygon.

(**A**) 60 (**B**) 70 (**C**) 80 (**D**) 90 (**E**) None of these

Solution:

$$\frac{S}{180} = \left(\frac{16}{24}\right)^2 = \frac{256}{576}.$$

Thus

$$S = 80.$$

Answer: C) 80

45. The sides of a triangle are 14, 16, and 18. Find the length of the longest altitude.

(**A**) $\frac{48\sqrt{5}}{7}$ (**B**) $4\sqrt{5}$ (**C**) $\frac{24\sqrt{5}}{7}$ (**D**) $8\sqrt{5}$ (**E**) None of these

Solution: Heron's formula:

$$s = 24, \quad A = \sqrt{24 \cdot 10 \cdot 8 \cdot 6} = 48\sqrt{5}.$$

Using smallest side 14 as base,

$$h = \frac{2A}{14} = \frac{48\sqrt{5}}{7}.$$

Answer: A) $\frac{48\sqrt{5}}{7}$

46. Suppose one bag of marbles contains 3 red marbles and 4 blue marbles, and a second bag contains 5 red marbles and 3 blue marbles. Suppose you choose a bag at random, then draw five marbles without replacement. What is the probability that all five marbles drawn are of the same color?

(**A**) 0 (**B**) $\frac{5}{56}$ (**C**) $\frac{1}{112}$ (**D**) $\frac{5}{144}$ (**E**) None of these

Solution:

Since the probability of drawing 5 marbles of the same color from the first bag is 0, the probability is

$$\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{5\cdot4\cdot3\cdot2\cdot1}{8\cdot7\cdot6\cdot5\cdot4}=\frac{1}{112}$$

Answer: C) $\frac{1}{112}$**.**

47. Suppose $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are both arithmetic progressions. If $a_{11}+b_{11}=100$ and $a_{16}+b_{16}=120$, determine the value of a_1+b_1.

(**A**) -35 (**B**) 12 (**C**) 60 (**D**) 89 (**E**) None of these

Solution:

Let d_a and d_b be the common differences of sequences $\{a_i\}_i, \{b_i\}_i$, resp., then

$$\begin{aligned}100 &= a_{11}+b_{11}\\ &= a_1+10d_a+b_1+10d_b\\ 120 &= a_{16}+b_{16}\\ &= a_1+15d_a+b_1+15d_b\\ 3\cdot100-2\cdot120 &= a_1+b_1.\end{aligned}$$

Answer: C) 60**.**

48. Suppose $-2.1<\log_{18}\frac{1}{432}<-2.099$. Which of these values is the best estimate of $\log_{24}18$?

(**A**) 0.476 (**B**) 0.667 (**C**) 0.909 (**D**) 1.133 (**E**) None of these

Solution:

$$\begin{aligned}\log_{18}\frac{1}{432} &= -\log_{18}432\\ &= -\log_{18}18\cdot24\\ &= -(1+\log_{18}24)\\ \log_{18}24 &\approx 1.1\\ \log_{24}18 &\approx \frac{1}{1.1}\\ &\approx 0.909\end{aligned}$$

Answer: C) 0.909**.**

49. Consider the ellipse centered at the origin O given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$. Let P be the focus of the ellipse with positive x-coordinate and let A_1 be the point on the ellipse with the same x-coordinate as P where A_1 is in the first quadrant. If $\frac{b}{a} = \frac{1}{5}$, determine $\tan(m\angle POA_1)$.

(**A**) $\frac{\sqrt{2}}{20}$ (**B**) $\frac{1}{20}$ (**C**) $\frac{\sqrt{6}}{60}$ (**D**) $\frac{\sqrt{2}}{40}$ (**E**) None of these

Solution: Let A_1 be at (c, d). We want $\frac{d}{c}$. Recall $c^2 = a^2 - b^2$ and $\frac{c^2}{a^2} + \frac{d^2}{b^2} = 1$. Hence,

$$d^2 = b^2(1 - \frac{c^2}{a^2}) = \frac{b^4}{a^2}.$$

This gives

$$\frac{d}{c} = \frac{d/a}{c/a} = \frac{b^2/a^2}{\sqrt{1 - \frac{b^2}{a^2}}} = \frac{1/25}{\sqrt{24}/5} = \frac{1}{10\sqrt{6}}.$$

Answer: C) $\frac{\sqrt{6}}{60}$

50. In polar coordinates, let A be the point $(3, 140°)$ and B be the point $(5, 170°)$. If d is the distance between A and B, and $d^2 = a - b\sqrt{c}$ where a, b, c are integers with c square-free, compute $a + b + c$.

(**A**) 49 (**B**) 50 (**C**) 51 (**D**) 52 (**E**) None of these

Solution: We use the Law of Cosines. Thus, $d^2 = 3^2 + 5^2 - 30\cos(30°) = 34 - 15\sqrt{3}$.

Answer: D) 52